Pluripotential theory and monge-ampère foliations

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Abstract A regular, rank one solution u of the complex homogeneous Monge-Ampère equation $(\partial\overline{\partial}u)^n = 0$ on a complex manifold is associated with the Monge-Ampère foliation, given by the complex curves along which u is harmonic. Monge-Ampère foliations find many applications in complex geometry and the selection of a good candidate for the associated Monge-Ampère foliation is always the first step in the construction of well behaved solutions of the complex homogeneous Monge-Ampère equation. Here, after reviewing some basic notions on Monge-Ampère foliations, we concentrate on two main topics. We discuss the construction of (complete) modular data for a large family of complex manifolds, which carry regular pluricomplex Green functions. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$. We then report on the problem of defining pluricomplex Green functions in the almost complex setting, providing sufficient conditions on almost complex structures, which ensure existence of almost complex Green pluripotentials and equality between the notions of stationary disks and of Kobayashi extremal disks, and allow extensions of known results to the case of non integrable complex structures.

1 Introduction

Pluripotential theory might be considered as the analogue in several complex variables of the potential theory associated with the Laplace operator. Indeed, it can be regarded as the potential theory in higher dimensions associated with the complex homogeneous Monge-Ampère equation.

For a function u of class $\mathscr{C}^2$ on an open set U of a complex manifold, the *complex (homogeneous) Monge-Ampère equation* is the equation on u of the form

$$(dd^c u)^n = (2i\partial\overline{\partial}u)^n = (2i)^n \underbrace{\partial\overline{\partial}u \wedge \ldots \wedge \partial\overline{\partial}u}_{n \text{ times}} = 0\,, \tag{1.1}$$

which, in local coordinates, is equivalent to

$$\det(u_{i\bar{k}}) dz^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n = 0 \quad \Longleftrightarrow \quad \det(u_{j\bar{k}}) = 0\,.$$

It is immediate to realize that, in complex dimension one, this equation reduces to the Laplace equation and it is well known that the Monge-Ampère operator may be meaningfully extended to much larger classes of functions. As (1.1) is invariant under biholomorphic maps, it is natural to expect that its solutions play a role of great importance in several complex variables as much as harmonic functions do in complex dimension one.

A distinctive feature of classical potential theory is the fact that harmonic functions, which are very regular, may be constructed maximizing families of (non regular) subharmonic functions. In fact, on one hand subharmonic functions are abundant and easy to construct since they do not need be very regular, on the other hand envelopes of suitable families of subharmonic functions are very regular and in fact harmonic. This construction scheme, systematized as *Perron method*, is based on maximum principle and it is both a basic tool and a key aspect of classical potential theory.

In higher dimension the peculiar role of subharmonic functions is played by the class of plurisubharmonic functions. As the suitable maximum principle holds also for the complex Monge-Ampère operator, Perron method has been successfully applied to construct solutions for the complex homogeneous Monge-Ampère equation satisfying boundary conditions. It turns out that the appropriate notion of maximality among plurisubharmonic functions is equivalent to be solution of (1.1) at least in a generalized sense. Here the analogy with the complex one dimensional case breaks down. The highly non linearity and the non ellipticity nature of equation (1.1) forces its solutions to be not regular even for very regular initial data. For instance, while positive results regarding the existence of solutions to the Dirichlet problem for (1.1) have been known since long time (see [6] for instance) it has been soon realized that it has at most $\mathscr{C}^{1,1}$ solutions even for the unit ball in $\mathbb{C}^n$ with real analytic datum on the boundary (see [4, 19]).

Potential theory in one complex variable plays a fundamental role in many areas of function theory, in particular in the uniformization theory of Riemann surfaces. For instance one singles out hyperbolic surfaces by the existence of Green functions, which are bounded above harmonic functions with a logarithmic singularity at one point and may be constructed by Perron method. It is natural to try and repeat the scheme in higher dimension replacing subharmonic functions with plurisubharmonic functions, their natural counterpart in higher dimension in order to define a natural generalization of Green function: the *pluricomplex Green function*. For instance for a domain $D \subset \mathbb{C}^n$, the pluricomplex Green function with logarithmic pole $z_0 \in D$ is defined by

$$G_D(z_0,z) = \sup\{u(z) \mid u \in PSH(D), u < 0, \limsup_{z\to z_0}[u(z) - \log\|z-z_0\|] < +\infty\}.$$

This is in complete analogy with the definition of Green function in complex dimension one and in fact $G_D(z_0,z)$ satisfies (1.1) on $D\setminus\{z_0\}$ (in the weak sense). It is known that pluricomplex Green functions exist for any hyperconvex domain in $\mathbb{C}^n$ (see [17]). On the other hand pluricomplex Green function does not satisfy basic properties which one may expect (and desire) to hold. For instance, even for a real analytic bounded strongly pseudoconvex domain D, the pluricomplex Green function $G_D(z_0,z)$ need not be of class $\mathscr{C}^2$ on $D\setminus\{z_0\}$, in general it fails to be symmetric, i.e., $G_D(w,z) \neq G_D(z,w)$, and one cannot even expect that $G_D(z_0,z)$ is subharmonic in z_0 ([3]).

There are several layers of understanding for the lack of regularity of solutions of complex homogeneous Monge-Ampère equation. The first is the most obvious one: non regularity as "defect of differentiability" and this motivates the need of understanding the equation in the weak sense. A second aspect is that non regularity is coupled with – and in many cases caused by – an excess or, rather, non constancy of degeneracy. More precisely, for a function u being solution of (1.1) is equivalent to the degeneracy of the form dd^cu, which is the same to ask that dd^cu has non trivial annihilator. In general the rank of the annihilator of dd^cu for a solution of (1.1) need not be the smallest possible (i.e., one) or constant. In some sense, this is a geometric aspect of the non regular behavior of (1.1).

The existence of regular solutions for the complex homogeneous Monge-Ampère equation with the least possible degeneracy defines a reach geometry: There exists a foliation in complex curves of the domain of existence of the solution such that the restriction of the solution to one of the leaf is harmonic. This geometric byproduct of existence of well behaved solutions, known as *Monge-Ampère foliation*, was explicitly studied for the first time by Bedford and Kalka [5]. Starting with the work of Stoll [52] (see [13] and [56] for alternative proofs), many of these ideas were exploited in questions of classification and characterization of special complex manifolds: See, for instance, [39, 40, 57, 28] for applications to the classification of circular domains and their generalizations and [47, 33, 53, 24, 14, 45] for the study of complexifications of Riemannian manifolds.

In fact, whenever such regular solutions exist, its construction starts with the determination of a suitable foliations. This is a well known fact, used also to provide examples of solutions with bad behaviors. In this regard, we mention the early example [2] and the most recent work by Lempert and Vivas [34] on the non-existence of regular geodesics joining points in the space of Kähler metrics (see also [10], in this volume).

On the other hand, in the seminal work of Lempert on convex domains ([31]), the existence of regular pluricomplex Green functions is related to a very good behavior of the Kobayashi metric for such domains, namely to the existence of a smooth foliation by Kobayashi extremal disks through any given point (see also for similar construction for pluri-Poisson kernels [11, 12]). In this case, the existence of a foliation by extremal disks is based on the equivalence between the notions of extremal and stationary disks, the latter being characterized as solution of manageable differential problem. Such link between Monge-Ampère equation, Kobayashi metric and stationary disks determines connections between many different problems and ways to approach them from various points of view.

Within this framework, we will report on two lines of research.

Following and simplifying ideas that go back to papers of Lempert for strictly convex domains ([32]) and of Bland-Duchamp for domains that are small deformations of the unit ball ([7, 8, 9]), we discuss the construction of (complete) modular data for a large family of complex manifolds, which carry regular pluricomplex Green functions. This class of manifolds naturally includes all smoothly bounded, strictly linearly convex domains and all smoothly bounded, strongly pseudoconvex circular domains of $\mathbb{C}^n$.

The modular data for this class of manifolds and, even more, the methods used, naturally suggest to ask similar questions for almost complex manifolds and to investigate the possibility of defining a useful notion of almost complex pluricomplex Green function. The generality of this setting poses new difficulties. The abundance of J-holomorphic curves, which is an advantage in many geometrical considerations, turns into a drawback when considering objects such as the Kobayashi metric. In particular, the notions of stationary and extremal disks are in general different ([21]). As for the construction of Green pluripotentials, it is necessary to cope with the behavior of plurisubharmonic functions in the non-integrable case, which may be is rather unexpected even for arbitrarily small deformations of the standard complex structure. Finally, the kernel distribution of (the natural candidate of) almost-complex Monge-Ampère operator, even if appropriate non-degenericity conditions are assumed, in principle are neither integrable, nor J-invariant. All this is in clear contrast with the classical setting and hence it cannot be expected that, for completely arbitrary non-integrable structures, one can reproduce the whole pattern of fruitful properties relating regular solutions of complex Monge-Ampère equations, foliations in disks and Kobayashi metric.

Nevertheless it is possible to determine sufficient conditions on the almost complex structure, which ensure the existence of almost complex Green pluripotential and the equality between the two notions of stationary disks and of extremal disks. The class of such structures is very large in many regards, in fact determined by a finite set of conditions (it is finite-codimensional) in an infinite dimensional space.

2 Domains of circular type and Monge-Ampère foliations

2.1 Circular domains and domains of circular type

Let $D \subset \mathbb{C}^n$ be a *complete circular domain*, i.e., such that $z \in D$ if and only if $\lambda z \in D$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. For simplicity, let us assume that it is smoothly bounded and strictly pseudoconvex. It is well known that any such domain D is completely determined by its *Minkowski functional* μ_D, which is the real-valued function

$$\mu_D : \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}\ , \qquad \mu_D(z) = \begin{cases} 0 & \text{if } z = 0 \\ \frac{1}{t_z} & \text{if } z \neq 0\ , \end{cases}$$

where $t_z = \sup\{\ t \in \mathbb{R}\ :\ tz \in D\ \}$. The square of the Minkowski functional

$$\rho_D : D \longrightarrow \mathbb{R}_{\geq 0}\ , \qquad \rho_D(z) = \mu_D(z)^2$$

is called *Monge-Ampère exhaustion of D* and satisfies some crucial properties. It can be considered as the modeling example of the Monge-Ampère exhaustions of the class of domains that we are going to analyze in the sequel.

One of the very first properties that can be inferred just from the definitions is the fact that ρ_D is always a map of the form

$$\rho_D(z) = G_D(z)\|z\|^2\ ,$$

where $G_D : D \setminus \{0\} \longrightarrow \mathbb{R}$ is a bounded function of class $\mathscr{C}^\infty$, which is constant on each complex line through the origin and hence identifiable with a $\mathscr{C}^\infty$ map $G_D : \mathbb{C}P^{n-1} \longrightarrow \mathbb{R}$, defined on the complex projective space $\mathbb{C}P^{n-1}$. It turns out that the exhaustion ρ_D provides full biholomorphic data for the moduli space of complete circular domains in the sense described in the following theorem.

Theorem 2.1 *[46] Two bounded circular domains D_1, $D_2 \subset \mathbb{C}^n$ are biholomorphic if and only if their Monge-Ampère exhaustions are such that*

$$\rho_{D_1} = \rho_{D_2} \circ A$$

for some $A \in \mathrm{GL}_n(\mathbb{C})$. Moreover, if we denote by $\mathscr{D}$ the set of biholomorphic classes of smoothly bounded complete circular domains, $\mathscr{D}^+ \subset \mathscr{D}$ the subset of biholomorphic classes of strictly pseudoconvex domains, by ω_{FS} the Fubini-Study 2-form of $\mathbb{C}P^{n-1}$, then

$$\mathscr{D} \simeq [\omega_{FS}]/\,\mathrm{Aut}(\mathbb{C}P^{n-1}) \qquad \text{and} \qquad \mathscr{D}^+ \simeq [\omega_{FS}]^+/\,\mathrm{Aut}(\mathbb{C}P^{n-1})\ ,$$

where we denote by $[\omega_{FS}]$ the cohomology class of ω_{FS} in $H^{1,1}(\mathbb{C}P^{n-1})$ and $[\omega_{FS}]^+ = \{$1-forms $\omega \in [\omega_{FS}]$ that are positive definite$\}$.

The description of the moduli spaces $\mathscr{D}$ and $\mathscr{D}^+$, given in this theorem, can be considered as a consequence of the following observation. Consider the unit ball $\mathbb{B}^n = \{\ \|z\|^2 < 1\ \}$ and a complete circular domain $D = \{\ \rho_D(z) = \mu_D^2(z) < 1\}$ with Monge-Ampère exhaustion $\rho_D(z) = G_D(z)\|z\|^2$. Both domains are blow downs of diffeomorphic disk bundles over $\mathbb{C}\mathbb{P}^{n-1}$ (Figure 1), which have

- "same" complex structure along fibers,
- "same" holomorphic bundle $\mathscr{H}$ normal to the fibers;

· “different” complex structures along $\mathscr{H}$.

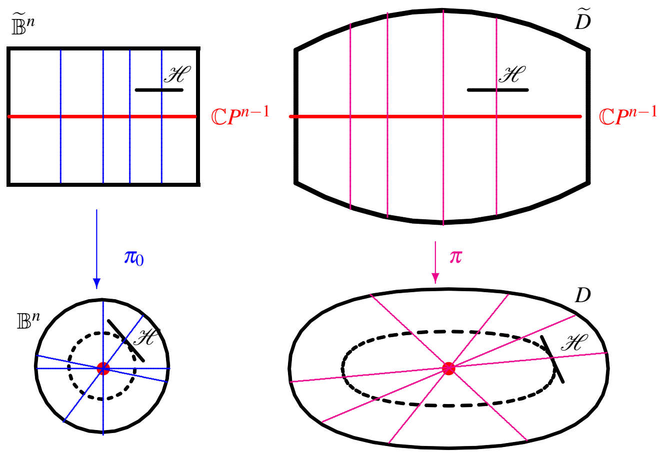

Figure 1

The differences between the complex structures on the normal holomorphic bundles $\mathscr{H}$ of $\mathbb{B}^n$ and D can be completely recovered from the map $G_D : \mathbb{C}P^{n-1} \longrightarrow \mathbb{R}$. This fact can be used to prove that the biholomorphic class of $[D]$ is completely determined (modulo actions of elements in $\mathrm{Aut}(\mathbb{C}P^{n-1})$) by the $(1,1)$-form $\omega = \omega_{FS} + \partial\bar{\partial}G_D$ and

$$[D] = [\mathbb{B}^n] \qquad \text{if and only if} \qquad \omega = \omega_{FS}$$

of course, up to actions of elements in $\mathrm{Aut}(\mathbb{C}P^{n-1})$.

Let us now consider the so-called *domains of circular type*, which are our main object of study for the first segment of these notes.

Before going into details, let us say a few words about notation. Since later we will have to deal with generalizations concerning almost complex manifolds it is useful to adopt notations that can be easily extended to the cases of non-integrable almost complex structures. With this purpose in mind, we recall that the familiar ∂- and $\overline{\partial}$-operators are related with the differential geometric operators d and $d^c = -J^* \circ d \circ J^*$ by the identities

$$d = \partial + \overline{\partial}\ , \qquad\qquad d^c = i(\overline{\partial} - \partial)\ .$$

and that $dd^c = -d^c d$ and $dd^c u = 2i\partial\overline{\partial}u$ for any $\mathscr{C}^2$ function $u : \mathscr{U} \subset M \longrightarrow \mathbb{R}$.

Let us now begin introducing the notion of manifolds of circular type. Here we give a definition which is slightly different from the original one, but nonetheless equivalent, as it follows from the results in [40].

Definition 2.2 ([40]) A pair (M,τ), formed by a complex manifold (M,J) of dimension n and a real valued function $\tau : M \longrightarrow [0,1)$, is called *(bounded) manifold of circular type with center x_o* if

i) $\tau : M \longrightarrow [0,1)$ is an exhaustion of M with $\{\tau = 0\} = \{x_o\}$ and satisfies the regularity conditions:

a) $\tau \in \mathscr{C}^0(M) \cap \mathscr{C}^\infty(\{\tau > 0\})$;
b) $\tau|_{\{\tau>0\}}$ extends smoothly over the blow up $\widetilde{M}$ at x_o of M;

ii) $$\begin{cases} 2i\partial\overline{\partial}\tau = dd^c\tau > 0\ , \\ \\ 2i\partial\overline{\partial}\log\tau = dd^c\log\tau \geq 0\ , \\ \\ (dd^c\log\tau)^n \equiv 0 \ \ \text{(Monge-Ampère Equation)}\ ; \end{cases}$$

iii) in some (hence *any*) system of complex coordinates $z = (z^i)$ centered at x_o, the function τ has a logarithmic singularity at x_o, i.e.

$$\log\tau(z) = \log\|z\| + O(1)\ .$$

A *domain of circular type with center x_o* is a pair (D,τ), given a relatively compact domain $D \subset M$ of a complex manifold M with smooth boundary, and an exhaustion $\tau : D \longrightarrow [0,1]$ smooth up to the boundary, such that (D,τ) is a manifold of circular type, i.e., satisfying the above conditions (i), (ii), (iii).

The simplest example of a domain of circular type is the unit ball $B^n \subset \mathbb{C}^n$, endowed with the standard exhaustion $\tau_o(z) = \|z\|^2$. In fact, *any* pair (D,ρ_D), formed by a strictly pseudoconvex, smoothly bounded, complete circular domain $D \subset \mathbb{C}^n$ and its Monge-Ampère exhaustion ρ_D, is a domain of circular type. To see this, it is enough to observe that ρ_D is strictly plurisubharmonic and that $\log\rho_D$ is plurisubharmonic, with harmonic restrictions on each (punctured) disk through the origin. The conditions (i) and (iii) are easily seen to be satisfied.

A much larger and interesting class of examples is given by the strictly (linearly) convex domains, whose properties, determined by the seminal work of Lempert, can be summarized as follows.

Theorem 2.3 ([31]) *Let $D \subset\subset \mathbb{C}^n$ be a smooth, bounded strictly (linearly) convex domain and denote by δ_D its Kobayashi distance and, for any given $x_o \in D$, by δ_{x_o} the function $\delta_{x_o} = \delta_D(x_o,\cdot)$. Then*

- $\delta_D \in \mathscr{C}^\infty(D \times D \setminus \mathrm{Diag})$*, where we denoted by* $\mathrm{Diag} = \{(z,z) \mid z \in D\}$*;*
- *the function $u = 2\log(\tanh\delta_{x_o})$ is in $\mathscr{C}^\infty(\overline{D} \setminus \{x_o\})$ and it is the unique solution of the problem*

$$\begin{cases} \det(u_{\mu\bar{\nu}}) = 0 & \textit{on}\ \ D \setminus \{x_o\}, \\ u_{|\partial D} = 0 & \textit{and}\ \ u(z) = 2\log\|z - x_o\| + O(1) \quad \textit{near}\ \ D \setminus \{x_o\}. \end{cases}$$

In fact, the pair (D, τ), with $\tau = (\tanh \delta_{x_o})^2$, is a domain of circular type with center x_o.

This theorem is indeed the result of a deep proof of geometric nature, which can be outlined as follows. Let x_o be a point of a smoothly bounded, strictly (linearly) convex domain $D \subset M$ and v a tangent vector in $T_{x_o}D$ of unit length w.r.t. to the infinitesimal Kobayashi metric κ_D. Lempert proved that there exists a unique complex geodesic

$$f_{\mathrm{v}} : \Delta \longrightarrow D\,, \qquad \Delta = \{\, |\zeta| < 1 \,\} \subset \mathbb{C}\,,$$

(i.e., a holomorphic map which is also an isometry between Δ, with its standard hyperbolic metric, and $\Delta^{(\mathrm{v})} = f_{\mathrm{v}}(\Delta) \subset D$, endowed with metric induced by the Kobayashi metric of D) such that

$$f_{\mathrm{v}}(0) = x_o\,, \qquad f'_{\mathrm{v}}(0) = \mathrm{v}\ .$$

He also shows that the complex geodesic f_{v} depends smoothly on the vector v and that the images of the punctured disk $\Delta \setminus \{0\}$,

$$f_{\mathrm{v}}(\Delta \setminus \{0\})\,, \qquad \mathrm{v} \in T_{x_o}M\,,$$

determine a smooth foliation of the punctured domain $D \setminus \{x_o\}$.

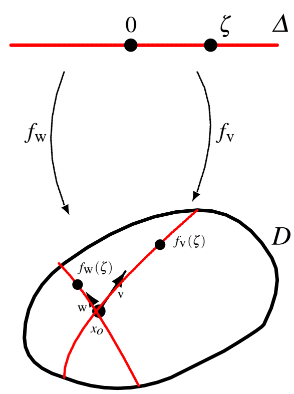

Figure 2

Using this and the fact that a holomorphic disc f_{v} is an isometry between $(\Delta \setminus \{0\}, \delta_\Delta)$ and $(f_{\mathrm{v}}(\Delta \setminus \{0\}), \delta_D) \subset (D, \delta_D)$, one gets that the map

$$u = 2\log(\tanh \delta_{x_o}) : \overline{D} \setminus \{x_o\} \longrightarrow \mathbb{R}$$

satisfies the equality

$$u(f_{\mathrm{v}}(\zeta)) = \log|\zeta|$$

for any complex geodesic f_{v}. In particular, $\tau(f_{\mathrm{v}}) = |\zeta|^2$. From these information, all other claims of the statement can be derived.

We remark that Lempert's result on existence of the foliation by extremal disks through a given point is based on the equivalence (for strictly convex domains) between the notions of extremal disks and of *stationary disks*, these being precisely the disks that realize the stationarity condition for the appropriate functional on holomorphic disks. We will come back to this point later on.

The problem of determining moduli for (pointed) strictly convex domains was addressed – and to a large extent solved – by Lempert, Bland and Duchamp in [32, 8, 7, 9].

The results in [32] can be summarized as follows. In that paper, it is proved that, for a given strictly convex domain $D \subset \mathbb{C}^n$ with a distinguished point $x_o \in D$ and for any given Kobayashi extremal disk $f_{\mathrm{v}} : \overline{\Delta} \longrightarrow \overline{D}$ with $f_{\mathrm{v}}(0) = x_o$ and $f'_{\mathrm{v}}(0) = \mathrm{v}$, there exists a special set of coordinates, defined on a neighborhood $\mathscr{U}$ of $f_{\mathrm{v}}(\partial\Delta) \subset \partial D$, in which the boundary ∂D admits a defining function $r : \mathscr{U} \longrightarrow \mathbb{R}$ of a special kind, called "normal form". The lower order terms of such defining functions in normal form (which can be considered as functions of the vectors $\mathrm{v} = f'_{\mathrm{v}}(0) \in T^{10}_{x_o}D$) determine biholomorphic invariants, which completely characterize the pointed domain (D, x_o) up to biholomorphic equivalences.

Bland and Duchamp's approach is quite different. Roughly speaking, they succeeded in constructing a complete class of invariants for any pointed, strictly convex domain (D, x_o) (and also for any pointed domain which is a sufficiently small deformation of the unit ball) using the Kobayashi indicatrix at x_o and a suitable "deformation tensor", defined on the holomorphic tangent spaces, which are normal to the extremal disks through x_o.

Lempert, Bland and Duchamp's results provide an excellent description of the moduli space of strictly convex domains, but they also motivate the following problems.

Problem 1. The moduli space of pointed convex domains appears to be naturally sitting inside a larger space. *Find the "right" family of domains corresponding to such larger space.*

Problem 2. The singular foliation of a circular domain by its stationary disks through the origin is very similar to the singular foliation of a strictly convex domain by stationary disks through a fixed point x_o. But there are also some crucial differences between such two situation: In the latter case, *any* point is center of a singular foliation, while in the former there apparently is only one natural choice for the center, the origin. *Determine an appropriate framework for "understanding" the possible differences between the "sets of centers" of the domains admitting (singular) stationary foliations*.

In the following two sections, we are going to discuss a simplification and a generalization of Bland and Duchamp's invariants, which brings to the following:

a) *the manifolds of circular type determine a moduli space, which naturally includes the moduli of strictly convex domains and on which Bland and Duchamp's invariants are in bijective correspondence;*
b) *this new construction of Bland and Duchamp's invariants determine a new setting, in which the sets of special points can be studied in a systematic way.*

2.2 *Homogeneous complex Monge-Ampère equation and Monge-Ampère foliations*

Let M be a complex manifold of dimension n and $u : M \longrightarrow \mathbb{R}$ be a function of class $\mathscr{C}^\infty$. In complex coordinates $(z^1, \ldots, z^n)$, we have that $dd^c u = 2i\partial\overline{\partial}u = 2i\sum u_{j\bar{k}} dz^j \wedge d\bar{z}^k$ and the plurisubharmonicity of a function u is equivalent to require that $dd^c u = 2i\partial\overline{\partial}u \geq 0$ or, in local coordinates, that $(u_{j\bar{k}}) \geq 0$.

The *complex (homogeneous) Monge-Ampère equation* is the equation on u of the form

$$(dd^c u)^n = (2i\partial\overline{\partial}u)^n = (2i)^n \underbrace{\partial\overline{\partial}u \wedge \ldots \wedge \partial\overline{\partial}u}_{n \text{ times}} = 0\,,$$

i.e., in local coordinates,

$$\det(u_{i\bar{k}}) dz^1 \wedge \ldots \wedge dz^n \wedge d\overline{z^1} \wedge \ldots \wedge d\bar{z}^n = 0 \quad \Big(\Longleftrightarrow\ \det(u_{j\bar{k}}) = 0\Big)\,.$$

Assume now that:

a) u is a smooth solution of a Monge-Ampère equation (that is $(dd^c u)^n = 0$);
b) u is plurisubharmonic (that is $dd^c u \geq 0$);
c) $\tau = e^u$ is *strictly* plurisubharmonic (that is $dd^c \tau > 0$).

We claim that (a), (b), (c) imply

$$(dd^c u)^n = 0 \ \text{ (by Monge-Ampère equation)} \qquad \textit{and} \qquad (dd^c u)^{n-1} \neq 0\,, \tag{2.1}$$

i.e., *the rank of $dd^c u$ is exactly $n-1$ at all points*. In fact, at every point p, the 2-form $dd^c u|_p$ is positive along directions in the holomorphic tangent space to level sets of u through p and $dd^c u|_p$ *has exactly $n-1$ positive eigenvalues and only one zero eigenvalue*.

Since it is useful for future developments, we give here some details of the proof. First of all, we observe that, by definitions,

$$e^{2u} dd^c u = \tau^2 dd^c \log\tau = \tau dd^c \tau - d\tau \wedge d^c \tau \tag{2.2}$$

and hence

$$(e^{2u})^n (dd^c u)^n = \tau^{2n} (dd^c \log \tau)^n = \tau^n (dd^c \tau)^n - n\tau^{n-1} (dd^c \tau)^{n-1} \wedge d\tau \wedge d^c \tau \; .$$

This implies that

$$(dd^c u)^n = 0 \qquad \text{if and only if} \qquad \tau (dd^c \tau)^n = n (dd^c \tau)^{n-1} \wedge d\tau \wedge d^c \tau$$

or, equivalently,

$$\det(u_{i\bar{k}}) = 0 \qquad \text{if and only if} \qquad \tau = -\sum_{\nu,\mu} \tau_{\overline{\nu}} \tau^{\overline{\nu}\mu} \tau_{\mu} \; , \quad (\tau^{\overline{\nu}\mu}) \overset{\text{def}}{=} (\tau_{\overline{\rho}\sigma})^{-1} \; . \quad (2.3)$$

Assume now that u satisfies the Monge-Ampère equation and consider the vector field Z in $T^{1,0}(M \setminus \{x_o\})$, determined by the condition

$$-\frac{i}{2} dd^c \tau(Z, \cdot) = \partial\overline{\partial}\tau(Z, \cdot) = \overline{\partial}\tau \; . \qquad (2.4)$$

Such vector field necessarily exists and is unique, because by assumptions the 2-form $dd^c\tau$ is non-degenerate (in fact, a Kähler metric). In coordinates, the vector fields Z is of the form

$$Z = \sum_{\mu,\nu} \tau_{\overline{\nu}} \tau^{\overline{\nu}\mu} \frac{\partial}{\partial z^{\mu}} \; . \qquad (2.5)$$

From (2.5), (2.4) and (2.3), it follows that

$$\partial\overline{\partial}\tau(Z, \overline{Z}) = \overline{\partial}\tau(\overline{Z}) = \tau \qquad (\text{ and hence also } \partial\tau(Z) = \tau \;) \; . \qquad (2.6)$$

Moreover, decomposing an arbitrary $(1,0)$-vector field V into a sum of the form

$$V = \lambda Z + W \; ,$$

where W is a $(1,0)$-vector field W tangent to the level sets $\{\tau = \text{const.}\}$ (and hence such that $d\tau(W) = 0$), using (2.2) and (2.6), we get

$$\begin{aligned} e^{2u} dd^c u(V, \overline{V}) &= |\lambda|^2 \tau dd^c \tau(Z, \overline{Z}) + 2\,\mathrm{Re}(\lambda \tau dd^c \tau(Z, \overline{W})) + \tau dd^c \tau(W, \overline{W}) \\ &\quad - (d\tau \wedge d^c \tau)(\lambda Z + W, \overline{\lambda}\overline{Z} + \overline{W}) \\ &= 2i|\lambda|^2 \tau^2 + 2\,\mathrm{Re}(\tau d\tau(\overline{W})) + \tau dd^c \tau(W, \overline{W}) - 2i|\lambda|^2 \tau^2 \\ &= \tau dd^c \tau(W, \overline{W}) \geq 0 \; , \end{aligned}$$

because the level sets of u coincide with the level sets of τ and these are strictly pseudoconvex. It follows that $dd^c u \geq 0$ is positively semi-definite and $\mathrm{Ann}\, dd^c u = \mathbb{C}Z$ on $M \setminus \{x_o\}$.

From these observations, we have that, for any $u : M \longrightarrow \mathbb{R}$ satisfying the above conditions (a), (b) and (c), the family of complex lines

$$\mathscr{Z} = \{ \mathscr{Z}_x \subset T_xM \ , \ x \neq x_o \ : \ \mathscr{Z}_x \text{ is the kernel of } \ dd^c u|_x \ \} \tag{2.7}$$

is actually a complex distribution of rank 1 with the following crucial properties:

– it is integrable (in fact, it coincides with $\mathrm{Ann}\, dd^c u$ and $dd^c u$ is a closed 2-form);
– its integral leaves are holomorphic curves (in fact, $dd^c u$ is a $(1,1)$-form).

The foliation $\mathscr{F}$ of the integral leaves of $\mathscr{Z}$ is called *Monge-Ampère foliation associated with u (or $\tau = e^u$).*

We point out that there exists a very simple criterion for determining whether a holomorphic curve is part of a leaf of $\mathscr{F}$: It suffices to observe that *the image $L(\Delta)$ of a holomorphic curve $L : \Delta \longrightarrow M$ is contained in an integral leaf of $\mathscr{Z}$ if and only if $u \circ L : \Delta \longrightarrow \mathbb{R}$ is a harmonic function.*

Various properties of the domains of circular type follows from the above conditions (a), (b), (c). We summarize them in the next theorem and we refer to [40, 41] for details and proofs.

Theorem 2.4 *Let (M,τ) be a manifold of circular type with center x_o and denote by $\pi_{x_o} : \widetilde{M} \longrightarrow M$ and $\pi_0 : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{B}^n$ the blow-ups of M and of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ at x_o and 0, respectively.*

There exists a diffeomorphism $\Psi : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{M}$ such that, for any $\mathrm{v} \in S^{2n-1} \subset T_0\mathbb{C}^n$, the map

$$f_{\mathrm{v}} : \Delta \longrightarrow M \ , \qquad f_{\mathrm{v}}(\zeta) = \pi_{x_o}\left(\Psi(\pi_0^{-1}(\zeta\, \mathrm{v}))\right)$$

is such that

a) it is proper, one-to-one and holomorphic;
b) its image $f_{\mathrm{v}}(\Delta)$ is (the closure of) a leaf of the Monge-Ampère foliation on $M \setminus \{x_o\}$ determined by τ;
c) it is the unique complex geodesic for the Kobayashi metric of M, passing through x_o and tangent to the vector $\mathrm{v} \in T_{x_o}M \simeq \mathbb{C}^n$.

Moreover, the map Ψ satisfies the additional property

$$(\tau \circ \Psi)|_{\widetilde{\mathbb{B}}^n \setminus \pi_0^{-1}(0)} = \|\cdot\|^2 \qquad (\ \|\cdot\| = \textit{Euclidean norm of } \mathbb{C}^n\ )\ .$$

In Figure 3, we try to schematize the properties of the map $\Psi : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{M}$ described in the above theorem

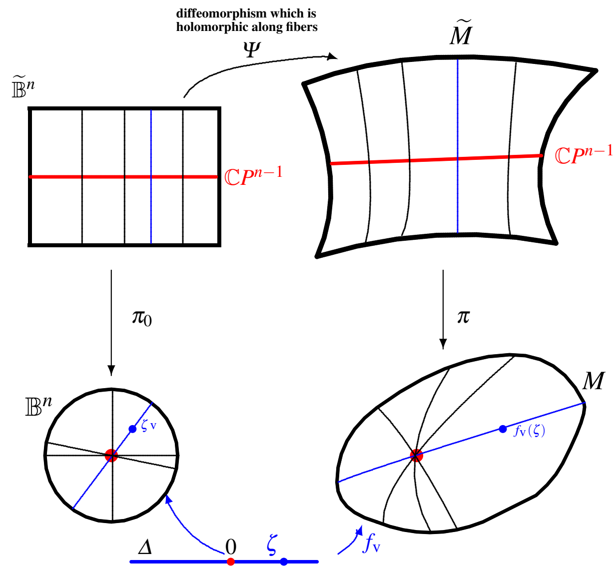

Figure 3

Rather than provide a complete argument, we sketch here the circle of ideas underlying these results. Under the assumptions of the theorem, there exist complex coordinates on a neighborhood $\mathscr{U} \subset M$ of x_o and centered at $x_o \simeq 0_{\mathbb{C}^n}$, in which τ assumes of the form

$$\tau(z) = h(z)\|z\|^2 + o(\|z\|^2) \tag{2.8}$$

for some $h : S^{2n-1} \subset T_{x_o}M \simeq \mathbb{C}^n \longrightarrow \mathbb{R}_*$ of class $\mathscr{C}^\infty$ and such that

$$h(\lambda z) = h(z) \qquad \text{for any } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1 \ .$$

Given the complex gradient $Z = \sum_{\mu,\nu} \tau_{\overline{\nu}} \tau^{\overline{\nu}\mu} \frac{\partial}{\partial z^\mu}$ of τ on $\mathscr{U} \setminus \{0\}$, consider the real vector fields

$$Y = \frac{1}{\sqrt{\tau}} \left(Z + \overline{Z}\right) \ , \qquad W = i\left(Z - \overline{Z}\right) \ ,$$

which determine the real and imaginary parts of the flow of of Z. By construction, Y and W are generators over $\mathbb{R}$ for the distribution (2.7), which we know it is integrable. Actually, using the Monge-Ampère equation satisfied by $u = \log \tau$, one can directly check that $[Y, W] = 0$.

Now, using (2.8) and once again the Monge-Ampère equation, one can show that near x_o the vector field Z is of the form

$$Z = \sum_{\mu,\nu} [z^\mu + G^\mu(z)] \frac{\partial}{\partial z^\mu} \qquad \text{for some } G^\mu(z) = O(\|z\|^2) \ , \tag{2.9}$$

so that the vector field $\widetilde{Z}$ on $(\Delta_\varepsilon \setminus \{0\}) \times (\mathbb{B}^n_r \setminus \{0\})$ with $\Delta_\varepsilon = \{ \zeta \in \mathbb{C} \ : \ |\zeta| < \varepsilon \}$, for a fixed $r > 1$ and $\varepsilon > 0$ sufficiently small, defined by

$$\widetilde{Z}(\lambda,z)=\sum_{\mu,\nu}\tau_{\overline{\nu}}(\lambda z)\tau^{\overline{\nu}\mu}(\lambda z)\frac{\partial}{\partial z^{\mu}}\ ,$$

extends in fact of class $\mathscr{C}^{\infty}$ on the whole open set $\widetilde{\mathscr{V}}=\Delta_{\varepsilon}\times\mathbb{B}^n_r$.

From previous remarks, the $\mathscr{C}^{\infty}$ vector fields

$$\widetilde{Y}=\frac{1}{\sqrt{\tau}}\left(\widetilde{Z}+\overline{\widetilde{Z}}\right)\ ,\qquad \widetilde{W}=i\left(\widetilde{Z}-\overline{\widetilde{Z}}\right)$$

satisfy

$$[\widetilde{Y},\widetilde{W}]=0$$

at all points of

$$\widetilde{\mathscr{V}}\setminus(\{0\}\times\mathbb{B}^n_r\cup\Delta_{\varepsilon}\times\{0_{\mathbb{C}^n}\})$$

and hence, by continuity, on the entire $\widetilde{\mathscr{V}}$.

Due to this, one can integrate such vector fields and, for any $\mathrm{v}\in\mathbb{B}^n_r$, construct a holomorphic map $\widetilde{f}_{\mathrm{v}}:\Delta_{\varepsilon}\longrightarrow\widetilde{\mathscr{V}}$ with $\widetilde{f}_{\mathrm{v}}(0)=(0,\mathrm{v})$ and such that

$$\widetilde{f}_{\mathrm{v}*}\left(\left.\frac{\partial}{\partial x}\right|_{\zeta}\right)=\left.\widetilde{Y}\right|_{\widetilde{f}_{\mathrm{v}}(\zeta)}$$

for any $\zeta\in\Delta_{\varepsilon}$. The collection of such holomorphic maps determine a map of class $\mathscr{C}^{\infty}$

$$\widetilde{F}:\Delta_{\varepsilon}\times S^{2n-1}\longrightarrow\widetilde{\mathscr{V}}\ ,\qquad \widetilde{F}(\zeta,\mathrm{v})=\widetilde{f}_{\mathrm{v}}(\zeta)\ .$$

By restriction on $\Delta_{\varepsilon}\times S^{2n-1}$, where $S^{2n-1}=\{\ z\in\mathbb{C}^n\ :\ \|z\|=1\ \}\subset\mathbb{B}^n_r$, and composing with the natural projection onto the blow up of $\mathscr{U}\simeq\mathbb{B}^n_r$,

$$\widetilde{\pi}:\widetilde{\mathscr{V}}=\Delta_{\varepsilon}\times\mathbb{B}^n_r\longrightarrow\widetilde{\mathbb{B}}^n_r\simeq\widetilde{\mathscr{U}}\ ,\qquad \widetilde{\pi}(\zeta,\mathrm{v})=([\mathrm{v}],\zeta\,\mathrm{v})\ ,$$

we get a smooth map

$$F:\Delta_{\varepsilon}\times S^{2n-1}\longrightarrow\mathscr{U}\subset M\ ,\qquad F=\pi\circ\widetilde{F}|_{\Delta_{\varepsilon}\times S^{2n-1}}\ .$$

This map extends uniquely to a smooth map $F:\Delta\times S^{2n-1}\longrightarrow M$ onto the whole complex manifold M, with the following properties:

- $\tau(F(\zeta,\mathrm{v}))=|\zeta|^2$ for any $\zeta\neq0$, so that $F(0,\mathrm{v})$ coincides with the corresponding point $([\mathrm{v}],\zeta\,\mathrm{v})$ of the singular set $E=\pi_0^{-1}(x_o)\simeq\mathbb{C}P^{n-1}$ for any $\mathrm{v}\in S^{2n-1}$;
- $F(\lambda\zeta,\mathrm{v})=F(\zeta,\lambda\,\mathrm{v})$ for any λ with $|\lambda|=1$;
- $f_{\mathrm{v}}=F(\cdot,\mathrm{v}):\Delta\longrightarrow\widetilde{M}$ is a biholomorphism between Δ and a (closure of) a leaf of the Monge-Ampère foliation determined by $u=\log\tau$ on $M\setminus\{x_o\}$;
- $\left.\widetilde{Z}\right|_{F(\zeta,\mathrm{v})}=\zeta f'_{\mathrm{v}}(\zeta)$ for any $\mathrm{v}\in S^{2n-1}$;
- $f'_{\mathrm{v}}(0)=\sqrt{h(\mathrm{v})}\,\mathrm{v}$, where h denotes the function in (2.8).

The map of Theorem 2.4 is precisely the map $\Psi:\widetilde{\mathbb{B}}^n\longrightarrow\widetilde{M}$ defined by

$$\Psi|_E = \mathrm{Id}_E \qquad \text{and} \qquad \Psi([\mathrm{v}], \zeta\,\mathrm{v}) = F\,(\zeta, \mathrm{v}) = f_{\mathrm{v}}(\zeta)$$

for any $([\mathrm{v}], \zeta\,\mathrm{v}) \in \widetilde{\mathbb{B}}^n \setminus E$ with $\mathrm{v} \in S^{2n-1}$. Using the above construction, one can directly check that Ψ is smooth.

It also turns out that the subset of $T_{x_o}M$ defined by

$$I_{x_o} = \{\ \mathrm{v} \in T_{x_o}M \simeq \mathbb{C}^n\ :\ h(\mathrm{v})\|\,\mathrm{v}\,\|^2 < 1\ \}$$

coincides with the *Kobayashi indicatrix of M at x_o*. In fact, this is a consequence of the following proposition ([41]).

Proposition 2.5 *For any $\mathrm{v} \in S^{2n-1} \subset \mathbb{C}^n \simeq T_{x_o}M$, the holomorphic disk $f_{\mathrm{v}} : \Delta \longrightarrow M$ is a Kobayashi extremal disk of M in the direction of v. This extremal disk is unique.*

Proof. To prove the first claim, we need to show that for any holomorphic map $g : \Delta \longrightarrow M$ with $g(0) = x_o$ and $g'(0) = t\,\mathrm{v}$ for some $t \in \mathbb{R}_{>0}$, we have that

$$t = \|g'(0)\| \leq \|f'_{\mathrm{v}}(0)\| = \sqrt{h(\mathrm{v})}\ .$$

For any such disk, consider the function

$$\ell : \Delta \longrightarrow \mathbb{R}\ , \qquad \ell(\zeta) = \log \tau(g(\zeta))\ .$$

It is subharmonic with $|\ell(\zeta)| \leq 0$ at all points and

$$|\ell(\zeta) - \log(|\zeta|)^2| = o(|z|)\ .$$

This means that $\log(|\zeta|^2)$ is a harmonic majorant for $\ell(\zeta)$ and that

$$\tau(g(\zeta)) \leq |\zeta|^2 = \tau\,(f_{\mathrm{v}}(\zeta))\ . \tag{2.10}$$

One can also check that the map

$$\widetilde{\tau} = \tau \circ \pi_{x_o} : \widetilde{M} \setminus \pi_{x_o}^{-1}(x_o) \longrightarrow \mathbb{R}$$

extends smoothly to a function $\widetilde{\tau} : \widetilde{M} \longrightarrow \mathbb{R}$ defined over the whole blow up $\widetilde{M}$ and that *the limit of $\tau_{\mu\bar{\nu}}(g(\zeta))$, for ζ tending to 0, exists and its value*

$$\lim_{\zeta \to 0} \tau(g(\zeta))_{\zeta\bar{\zeta}} = \widetilde{\tau}_{\mu\overline{\nu}}([g'(0)], 0)\mathrm{v}^\mu \overline{\mathrm{v}}^\nu = \widetilde{\tau}_{\mu\overline{\nu}}([\mathrm{v}], 0)\,\mathrm{v}^\mu\,\overline{\mathrm{v}}^\nu$$

depends only on the element $[\mathrm{v}] \in E = \mathbb{C}P^{n-1}$.

From (2.10), we have that $\tau(g(\zeta)) = r(\zeta)|\zeta|^2$ for some smooth function $0 \leq r \leq 1$ and that

$$\tau(g(\zeta))_{\zeta\bar{\zeta}} = \tau_{\mu\bar{\nu}}(g(\zeta))g'^{\mu}(\zeta)\overline{g'^{\nu}(\zeta)} = r_{\zeta\bar{\zeta}}(\zeta)|\zeta|^2 + r_{\zeta}(\zeta)\zeta + r_{\bar{\zeta}}(\zeta)\overline{\zeta} + r(\zeta)\ .$$

All this implies that

$$t^2\widetilde{\tau}_{\mu\overline{\nu}}([\mathrm{v}],0)\,\mathrm{v}^{\mu}\,\overline{\mathrm{v}}^{\nu} = \lim_{\zeta\to 0}\tau_{\mu\bar{\nu}}(g(\zeta))g'^{\mu}(\zeta)\overline{g'^{\nu}(\zeta)} = r(0) \le 1 \; .$$

On the other hand, recalling that $\tau(f_{\mathrm{v}}(\zeta)) = |\zeta|^2$ and that $f'_{\mathrm{v}}(0) = \sqrt{h(\mathrm{v})}\,\mathrm{v}$, we have

$$1 = \tau(f_{\mathrm{v}}(0))_{\zeta\bar{\zeta}} = \lim_{\zeta\to 0}\tau_{\mu\bar{\nu}}(f_{\mathrm{v}}(\zeta))f'^{\mu}_{\mathrm{v}}(\zeta)\overline{f'^{\nu}_{\mathrm{v}}(\zeta)} = h(\mathrm{v})\widetilde{\tau}_{\mu\overline{\nu}}([\mathrm{v}],0)\mathrm{v}^{\mu}\overline{\mathrm{v}}^{\nu} \; ,$$

from which it follows that

$$\widetilde{\tau}_{\mu\overline{\nu}}([\mathrm{v}],0)\,\mathrm{v}^{\mu}\,\overline{\mathrm{v}}^{\nu} = \frac{1}{h(\mathrm{v})} = \frac{1}{\|f'_{\mathrm{v}}(0)\|^2} \qquad \text{so that} \qquad \|g'(0)\| = t \le \left\|f'_{\mathrm{v}}(0)\right\| \; .$$

It now remains to check the uniqueness of such extremal disk. First of all, notice that if $g : \Delta \longrightarrow M$ is a holomorphic disk with $g(0) = x_o$ and $g'(0) = \mathrm{v} = f'_{\mathrm{v}}(0)$, and $r(\zeta) \le 1$ is the function defined above, such that $\tau(g(\zeta)) = r(\zeta)|\zeta|^2$, then $r(0) = 1$. Moreover:

a) The function $\log r$ is subharmonic. In fact

$$\Delta \log r = \Delta \log \tau \circ g - \Delta \log|\zeta|^2 = \Delta \log \tau \circ g \ge 0 \; .$$

b) The function $\log r$ is always less or equal to 0.

By Maximum Principle, conditions (a), (b) and the equality $r(0) = 1$ imply that $r(\zeta) = 1$ and $\tau(g(\zeta)) = |\zeta|^2$ at all points. A little additional computation shows that $g(\Delta)$ is necessarily included in a leaf of the Monge-Ampère foliation determined by $u = \log\tau$. Using the fact that $\log\tau|_{g(\Delta)}$ is harmonic, one concludes that there is only one possibility for g, namely $g = f_v$. □

This proposition concludes our outline of the ideas behind the proof of Theorem 2.4 and the various properties of the diffeomorphism $\Psi : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{M}$. There is however another very important information on the diffeomorphism Ψ which we want to point out.

Assume that M is a smoothly bounded, relatively compact domain in a larger complex manifold N and that the exhaustion τ extends smoothly up to the boundary ∂M in such a way that $dd^c\tau > 0$ also at the boundary points of M. In this case, one can check that the map Ψ extends smoothly up to the closures of the blow-ups

$$\Psi : \overline{\widetilde{\mathbb{B}}^n} \longrightarrow \overline{\widetilde{M}}$$

and, consequently, all holomorphic disk $f_{\mathrm{v}} : \Delta \longrightarrow M$ extend to smooth maps $f_{\mathrm{v}} : \overline{\Delta} \longrightarrow \overline{M}$. This extendability property will turn out to be important for the construction of the normal forms and Bland and Duchamp's invariants that we are going to present in the next section.

Furthermore, the fact that such extremal disks are smoothly attached to the boundary determine a crucial relation between these disks and the geometry of the boundary. It turns out that *for any* $\mathrm{v} \in S^{2n-1} \subset T_{x_o}M$*, the corresponding holomorphic disk* $f_{\mathrm{v}} : \overline{\Delta} \longrightarrow \overline{M}$ *is stationary*, i.e., (see [31] for the original definition) there exists a holomorphic map

$$\widetilde{f}_{\mathrm{v}} : \overline{\Delta} \longrightarrow T^*M$$

such that

i) $\widehat{\pi} \circ \widehat{f}_{\mathrm{v}}(\zeta) = f_{\mathrm{v}}(\zeta)$ for any $\zeta \in \Delta$, where $\widehat{\pi} : T^*M \longrightarrow M$ is the standard projection;
ii) for any $\zeta \in \partial\Delta$, the 1-form $\zeta^{-1} \cdot \widehat{f}_{\mathrm{v}}(\zeta) \in (T^*_{f_{\mathrm{v}}(z)}M)^{10}$ is non-zero and belongs to the conormal bundle of ∂M (i.e., vanishes on the tangent of ∂M).

In fact, since the restrictions of u on the leaves of the Monge-Ampère equation are harmonic, one can immediately check that, for any given disk $f_{\mathrm{v}} : \overline{\Delta} \longrightarrow M$, the required map $\widetilde{f}_{\mathrm{v}} : \overline{\Delta} \longrightarrow (T^*_{f_{\mathrm{v}}(z)}M)^{10}$ is given by

$$\widetilde{f}_{\mathrm{v}}(\zeta) = \zeta \, \partial u|_{f_{\mathrm{v}}(\zeta)} .$$

We conclude this section with the following result by W. Stoll ([52]; see also the alternative proofs in [13, 56]), which was essentially one of the starting points of the geometrical applications of the theory of Monge-Ampère foliation theory.

Theorem 2.6 *(Stoll) Let M be a complex manifold complex manifold of dimension n. Then there exists a $\mathscr{C}^\infty$ exhaustion $\tau \colon M \to [0,1)$ such that*

$$(1)\ dd^c\tau > 0 \text{ on } M \qquad\qquad (2)\ (dd^c \log\tau)^n = 0 \text{ on } M \setminus \{\tau = 0\}$$

if and only if there exists a biholomorphic map $F \colon M \to \mathbb{B}^n = \{Z \in \mathbb{C}^n \mid \|Z\|^2 < 1\}$ with $\tau(F(z)) = \|z\|^2$

Two key remarks allow to prove Stoll's theorem using Theorem 2.4:

i) the minimal set $\{\tau = 0\}$ of τ reduces to a singleton $\{x_o\}$ so that (M,τ) is a manifold of circular type with center x_o;
ii) the map $\Phi : \mathbb{B}^n \longrightarrow M$, defined by requiring that the following diagram commutes (here π_0, π_{x_o} are blow down maps and $\Psi : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{M}$ is the map given in Theorem 2.4),

$$\begin{tikzcd} \widetilde{\mathbb{B}}^n \arrow[r, "\Psi"] \arrow[d, "\pi_0"'] & \widetilde{M} \arrow[d, "\pi_{x_o}"] \\ \mathbb{B}^n \arrow[r, "\Phi"] & M \end{tikzcd} \tag{2.11}$$

is a smooth diffeomorphism (even at the origin!).

To prove (i), one has first to observe that, as consequence of a result of Harvey and Wells [26], the level set $\{\tau = 0\}$ is totally real, compact and discrete (and hence finite). Then the conclusion follows by an argument of Morse theory: M is connected and retracts onto $\{\tau = 0\}$ along the flow of the vector field Y, which turns out to be the gradient of $\sqrt{\tau}$ with respect to the Kähler metric g determined by the Kähler form $dd^c\tau$.

Property (ii) follows from the fact that it is possible to show that Φ is a reparametrization of the exponential map at x_o of the metric g and it is therefore smooth at x_o.

By a classical result of Hartogs on series of homogeneous polynomials, from the fact that Φ is smooth and holomorphic along each disk through the origin, it follows that Φ is holomorphic. The fact that $\tau(\Phi(z)) = \|z\|^2$ is a consequence of Theorem 2.4 and the commutativity of the diagram (2.11).

3 Normal forms and deformations of CR structures

3.1 The normal forms of domains of circular type

In this section we constantly use the following notation:

- $\mathbb{B}^n$ is the unit ball of $\mathbb{C}^n$, centered at the origin;
- J_{st} is the standard complex structure of $\mathbb{C}^n$;
- $\pi : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{B}^n$ is the blow up of $\mathbb{B}^n$ at the origin;
- $\tau_o : \mathbb{B}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the standard Monge-Ampère exhaustion of $\mathbb{B}^n$, i.e., $\tau_o = \|\cdot\|^2$;
- $u_o : \mathbb{B}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the function $u_o = \log \tau_o^2$.

We will also denote by $\mathscr{Z} = \bigcup_{x \in \mathbb{B}^n \setminus \{0\}} \mathscr{Z}_x$ and $\mathscr{H} = \bigcup_{x \in \mathbb{B}^n \setminus \{0\}} \mathscr{U}_x$ the distributions on $\mathbb{B}^n \setminus \{0\}$, determined by the following subspaces of tangent spaces:

$$\mathscr{Z}_x = \text{0-eigenspace of the matrix } \left(u_{o\,j\bar{k}}\right)\Big|_x \subset T_x\mathbb{B}^n\ , \tag{3.1}$$

$$\mathscr{H}_x = \text{orthogonal complement of } \mathscr{Z}_x \text{ in } T_x\mathbb{B}^n \text{ (w.r.t Euclidean metric) .} \tag{3.2}$$

Notice that, for any point $x \in \mathbb{B}^n \setminus \{0\}$, the space $\mathscr{Z}_x$ is nothing else but the tangent space of the straight complex lines of $\mathbb{C}^n$, passing through x and 0 and that the (closures of) integral leaves of $\mathscr{Z}$ are the straight disks through the origin. The distributions $\mathscr{Z}$ and $\mathscr{H}$ will be called *radial* and *normal distributions*, respectively. They are both smoothly extendible at all points of the blow up $\widetilde{\mathbb{B}}^n$.

The main ingredients of this section consist of the objects introduced in the following definition.

Definition 3.1 A complex structure J on $\widetilde{\mathbb{B}}^n$ is called *L-complex structure* if and only if

i) the distributions $\mathscr{Z}$ and $\mathscr{H}$ are both J-invariant;
ii) $J|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}}$ (i.e., J and J_{st} differ only for their actions on $\mathscr{H}$!);
iii) there exists a smooth homotopy $J(t)$ of complex structures, all of them satisfying (i) and (ii), with $J(0) = J_{\mathrm{st}}$ and $J(1) = J$.

A complex manifold $M = (\mathbb{B}^n, J)$, which is the blow-down at 0 of a complex manifold of the form $(\widetilde{\mathbb{B}}^n, \widetilde{J})$, for some L-complex structure $\widetilde{J}$, is called *manifold of circular type in normal form*.

The crucial property of this class of manifolds is the following:

Proposition 3.2 *If $M = (\mathbb{B}^n, J)$ is a complex manifold, which is blow-down at* 0 *of a complex manifold of the form $(\widetilde{\mathbb{B}}^n, \widetilde{J})$, for some L-complex structure $\widetilde{J}$, the pair $(M = (\mathbb{B}^n, J), \tau_o)$ is a manifold of circular type with center $x_o = 0$.*

The proof essentially consists of checking that the exhaustion $\tau_o : \mathbb{B}^n \longrightarrow \mathbb{R}_{\geq 0}$ is strictly plurisubharmonic w. r. t. the (non-standard) complex structure J, i.e., $dd^c_J \tau_o > 0$. Here, "d^c_J" is the operator $d^c_J = -J \circ d \circ J$ and is in general different from the usual operator $d^c = -J_{\mathrm{st}} \circ d \circ J_{\mathrm{st}} = i(\partial - \overline{\partial})$ determined by J_{st}.

Due to J-invariance, the radial and normal distributions $\mathscr{Z}$, $\mathscr{H}$ are not only orthogonal w.r.t. the Euclidean metric but also w.r.t. the J-invariant 2-form $dd^c_J \tau_o$. Moreover, since $J|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}}$, we have that

$$dd^c_J \tau_o|_{\mathscr{Z} \times \mathscr{Z}} = dd^c \tau_o|_{\mathscr{Z} \times \mathscr{Z}} > 0 \ .$$

Therefore, what one really needs to check is that $dd^c_J \tau_o|_{\mathscr{H} \times \mathscr{H}} > 0$. By construction, for any $x \in \mathbb{B}^n \setminus \{0\}$, the subspace $\mathscr{H}_x \subset T_x \mathbb{B}^n$ coincides with the J-holomorphic tangent space of the sphere $S_c = \{ \tau_o = c \}$ with $c = \tau_o(x)$. Indeed, it is also the $J(t)$-holomorphic tangent space of S^{2n-1} *for any complex structure $J(t)$ of the isotopy between J_{st} and J*. It follows that the restriction

$$dd^c_{J(t)} \tau_o \Big|_{\mathscr{H}_x \times \mathscr{H}_x}$$

is the Levi forms of S_c at x for any complex structure $J(t)$. On the other hand, the distribution $\mathscr{H}|_{S_c} \subset TS_c$ is a contact distribution (it is the standard contact distribution of S_c) and, consequently, all such Levi forms are non-degenerate. Since $dd^c_{J(t)} \tau_o|_{\mathscr{H}_x \times \mathscr{H}_x}$ is positively when $t = 0$ and the complex structure is $J(0) = J_{\mathrm{st}}$, by continuity, the Levi forms $dd^c_{J(t)} \tau_o \Big|_{\mathscr{H}_x \times \mathscr{H}_x}$ are positively defined for any t and in particular when the complex structure is $J = J(1)$. This shows that $dd^c_J \tau_o|_{\mathscr{H}_x \times \mathscr{H}_x} > 0$ at all points as we needed.

By previous proposition, the manifolds "in normal form" constitute a very large family of examples manifolds of circular type, with an exhaustion $\tau_o = \| \cdot \|$ which is particularly simple. Moreover, the key result on such manifolds is the following.

Theorem 3.3 (Existence and uniqueness of normalizing maps) *For each manifold of circular type (M,J), with exhaustion τ and center x_o, there exists a biholomorphism $\Phi : (M,J) \longrightarrow (\mathbb{B}^n, J')$ to a manifold in normal form $(\mathbb{B}^n, J')$ with*

a) $\Phi(x_o) = 0$ and $\tau = \tau_o \circ \Phi$;
b) Φ maps the leaves of the Monge-Ampère foliation of M into the straight disks through the origin of $\mathbb{B}^n$.

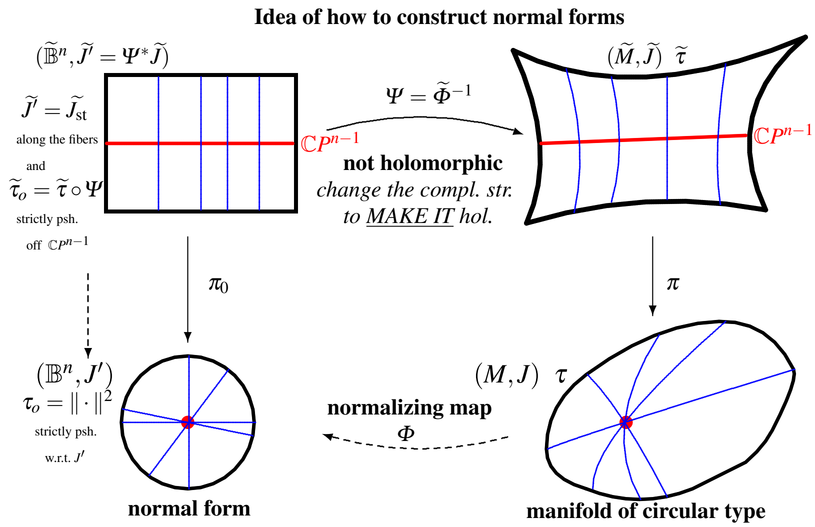

Figure 4

Any biholomorphism $\Phi : (M,J) \longrightarrow (\mathbb{B}^n, J')$, satisfying the conditions (a) and (b), is called *normalizing map for the manifold* M. The lifted map $\widetilde{\Phi} : \widetilde{M} \longrightarrow \widetilde{\mathbb{B}}^n$ between the blow ups at x_o and $\Phi(x_o) = 0$ is nothing but the inverse

$$\widetilde{\Phi} = \Psi^{-1}$$

of the diffeomorphism $\Psi : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{M}$, described in Theorem 2.4. The L-complex structure J' on $\mathbb{B}^n$ is constructed in such a way that the corresponding complex structure $\widetilde{J}'$ on the blow-up $\widetilde{\mathbb{B}}^n$ coincides with the complex structure on $\widetilde{\mathbb{B}}^n$, obtained by push-forwarding the complex structure $\widetilde{J}$ of $\widetilde{M}$ onto $\widetilde{\mathbb{B}}^n$, i.e.,

$$\widetilde{J}' \overset{\text{def}}{=} \Phi_*(\widetilde{J}) \ .$$

Notice also that two normalizing maps, related with the same exhaustion τ and same center x_o, differ only by their action on the leaf space of the Monge-Ampère foliation. In fact, it turns out that the class $\mathscr{N}(M)$ of all normalizing maps, determined by all its exhaustions of M (which might correspond to distinct centers), is naturally parameterized by a suitable subset of $\mathrm{Aut}(\mathbb{B}^n)$, which includes $\mathrm{Aut}(\mathbb{B}^n)_0 = \mathrm{U}_n$. We will discuss this point in more details in §3.3.

3.2 Normal forms and deformations of CR structures

Let $(\mathbb{B}^n, J)$ be a manifold of circular type in normal form, which is blow down of $(\widetilde{\mathbb{B}}^n, J)$ for an L-complex structure J (for simplicity, we will use the same symbol for the two complex structures). By definitions, J is completely determined by the restriction $J|_{\mathscr{H}}$ and such restriction is uniquely determined by the corresponding J-anti-holomorphic subbundle $\mathscr{H}^{01}_J \subset \mathscr{H}^{\mathbb{C}}$, formed by the $(-i)$-eigenspaces $\mathscr{H}^{01}_{Jx} \subset \mathscr{H}^{\mathbb{C}}_x$, $x \in \widetilde{\mathbb{B}}^n$ of the $\mathbb{C}$-linear maps $J_x : \mathscr{H}^{\mathbb{C}}_x \longrightarrow \mathscr{H}^{\mathbb{C}}_x$.

If we denote by $\mathscr{H}^{01} \subset \mathscr{H}^{\mathbb{C}}$ the J_{st}-anti-holomorphic subbundle, given by the standard complex structure J_{st}, in almost all cases the L-complex structure J can be recovered by the tensor field

$$\phi_J \in (\mathscr{H}^{01})^* \otimes \mathscr{H}^{10} = \bigcup_{x \in \widetilde{\mathbb{B}}^n} \mathrm{Hom}(\mathscr{H}^{01}_x, \mathscr{H}^{10}_x) \ ,$$

defined by the condition

$$\mathscr{H}^{01}_{Jx} = \{ \ \mathrm{v} = \mathrm{w} + \phi_J(\mathrm{w}) \ , \ \mathrm{w} \in \mathscr{H}^{01} \ \} \ . \tag{3.3}$$

This tensor field ϕ_J is called *deformation tensor of J*.

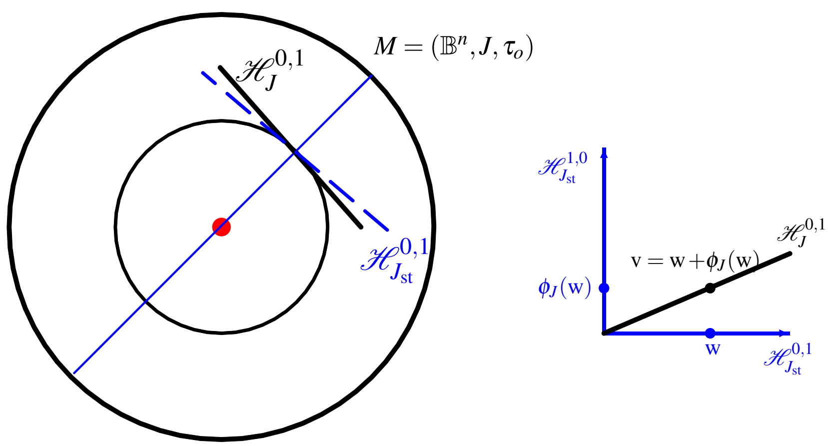

Figure 5

We however point out that the existence of a deformation tensor ϕ_J, associated with a complex structure J, which satisfies only (i) and (ii) of Definition 3.1, is a priori not always granted: It exists whenever, at any $x \in \widetilde{\mathbb{B}}^n$, the natural projection

$$p : \mathscr{H}^{\mathbb{C}}_x = \mathscr{H}^{10}_x + \mathscr{H}^{01}_x \longrightarrow \mathscr{H}^{01}_x$$

determines a linear isomorphism $p|_{\mathscr{H}^{01}_{Jx}} : \mathscr{H}^{01}_{Jx} \xrightarrow{\sim} \mathscr{H}^{01}_{x}$. This is an "open" condition, meaning that if J can be represented by a deformation tensor ϕ_J, then also the sufficiently close complex structures J', satisfying (i) and (ii), are representable by deformation tensors.

However, as we will shortly see, the existence of a deformation tensor for L-complex structures is also a "closed" condition and hence *any L-complex structure is represented by a deformation tensor*.

By these remarks, we have that the biholomorphic classes of domains of circular type are in natural correspondence with the deformation tensors of L-complex structures on $\widetilde{\mathbb{B}}^n$. It is therefore very important to find an efficient characterization of the tensor fields $\phi = (\mathscr{H}^{01})^* \otimes \mathscr{H}^{10}$ that correspond to L-complex structures, by a suitable set of intrinsic properties. This problem was solved by Bland and Duchamp in [7, 8, 9] via a suitable adaptation of the theory of deformations of complex structures (see [30] for a classical introduction).

In those papers, Bland and Duchamp were concerned with strictly linearly convex domains in $\mathbb{C}^n$ that are small deformations of the unit ball $\mathbb{B}^n$. To any such domain D, they associated a deformation tensor field ϕ_D on $\partial\mathbb{B}^n$, which, in our terminology, is the restriction to $\partial\mathbb{B}^n$ of the deformation tensor of the L-complex structure J of a normal form.

The arguments of Bland and Duchamp extend very naturally to all cases of our more general context and bring to the characterization of L-complex structures, which we are now going to describe.

First of all, notice that the holomorphic and anti-holomorphic distributions $\mathscr{H}^{10}$, $\mathscr{H}^{01}$ can be (locally) generated by vector fields $X^{1,0} \in H^{1,0}$, $Y^{0,1} \in \mathscr{H}^{0,1}$ such that

$$\hat{\pi}_* \left([X^{1,0}, Y^{0,1}]\right) = [\hat{\pi}_*(X^{1,0}), \hat{\pi}_*(Y^{0,1})] = 0 \ ,$$

where $\widehat{\pi} : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{C}^{n-1}$ is the natural fibering over the exceptional set $\mathbb{C}P^{n-1}$ of the blow up $\widetilde{\mathbb{B}}^n$. Let us call the vector fields of this kind *holomorphic* (resp. *anti-holomorphic*) *vector fields* of $\mathscr{H}^{\mathbb{C}}$.

Now, consider the following operators (see e.g. [30]) (here, we denote by $(\cdot)_{\mathscr{H}^{\mathbb{C}}} : T^{\mathbb{C}}\widetilde{\mathbb{B}}^n \longrightarrow \mathscr{H}^{\mathbb{C}}$ the natural projection, determined by the decomposition $T^{\mathbb{C}}\widetilde{\mathbb{B}}^n = \mathscr{Z}^{\mathbb{C}} + \mathscr{H}^{\mathbb{C}}$):

$$\bar{\partial}_b : H^{0,1*} \otimes H^{1,0} \to \Lambda^2 H^{0,1*} \otimes H^{1,0} \ ,$$

$$\bar{\partial}_b \alpha(X,Y) \stackrel{\text{def}}{=} [X, \alpha(Y)]_{\mathscr{H}^{\mathbb{C}}} - [Y, \alpha(X)]_{\mathscr{H}^{\mathbb{C}}} - \alpha([X,Y]) \ , \tag{3.4}$$

and

$$[\cdot,\cdot] : \left(H^{0,1*} \otimes H^{1,0}\right) \times \left(H^{0,1*} \otimes H^{1,0}\right) \longrightarrow \Lambda^2 H^{0,1*} \otimes H^{1,0} \ ,$$

$$[\alpha,\beta](X,Y) \stackrel{\text{def}}{=} \frac{1}{2}\left([\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)]\right) \tag{3.5}$$

for any pair of holomorphic and anti-holomorphic vector fields X, Y of $\mathscr{H}$. We then have the following:

Theorem 3.4 ([42]) *Let J be an L-complex structure on $\widetilde{\mathbb{B}}^n$ that admits a deformation tensor ϕ (in fact, J is an arbitrary L-complex structure). Then:*

i) $dd^c\tau_o(\phi(X),Y)+dd^c\tau_o(X,\phi(Y))=0$ *for any pair X, Y of vector fields in $\mathscr{H}^{0,1}$;*
ii) $\bar{\partial}_b\phi+\frac{1}{2}[\phi,\phi]=0$*;*
iii)$\mathscr{L}_{Z^{0,1}}(\phi)=0$.

Conversely, any tensor field $\phi\in H^{0,1*}\otimes H^{1,0}$ that satisfies (i) - (iii) is the deformation tensor of an L-complex structure.

In addition, an L-complex structure J, associated with a deformation tensor ϕ, is so that $(\mathbb{B}^n,J,\tau_o)$ is a manifold of circular type if and only if

iv) $dd^c\tau_o(\phi(X),\overline{\phi(X)})<dd^c\tau_o(\bar{X},X)$ *for any* $0\neq X\in H^{0,1}$.

For the proof we refer directly to [42]. Here, we only point out that the conditions (i) - (iii) comes out from the request of integrability for the almost complex structure J, coinciding with J_{st} on $\mathscr{Z}$ and with anti-holomorphic distribution $\mathscr{H}^{01}_J$ determined by (3.3).

Remark 3.5 Condition (iv) of previous theorem can be interpreted as an a-priori estimate for the deformation tensor ϕ: It gives an "upper bound" for the norm of ϕ w.r.t. to the Kähler metric $dd^c\tau_o$. It is this property that makes the representability of an L-complex structure by a deformation tensor a "closed" condition and that it implies the existence of a deformation tensor for *any* L-complex structure, as previously pointed out.

Consider now a local trivialization of the line bundle $\widetilde{\pi}:\widetilde{\mathbb{B}}^n\longrightarrow\mathbb{C}P^{n-1}$, which represents the points $z=([v],v)\in\widetilde{\pi}^{-1}(\mathscr{U})$ of some open subset $\mathscr{U}\subset\mathbb{C}P^{n-1}$ by pairs $(\mathrm{w},\zeta)\in S^{2n-1}\times\Delta$ with

$$\mathrm{w}=\frac{\mathrm{v}}{\|\mathrm{v}\|}\in S^{2n-1}\qquad\text{and}\qquad\mathrm{v}=\zeta\,\mathrm{w}\ .$$

Condition (iii) of Theorem 3.4 implies that the restriction $\phi_J|_{\pi^{-1}(\mathscr{U})}$ of the deformation tensor of J is of the form

$$\phi_J=\sum_{k=0}^{\infty}\phi_J^{(k)}(\mathrm{w},\zeta)=\sum_{k=0}^{\infty}\phi_J^k(\mathrm{w})\zeta^k\ ,$$

where each $\phi_J^{(k)}(\mathrm{w},\zeta)\overset{\text{def}}{=}\phi_J^k(\mathrm{w})\zeta^k$ is a tensor in $\left(\mathscr{H}^{01*}\otimes\mathscr{H}^{10}\right)\big|_{([\mathrm{w}],\zeta\,\mathrm{w})}$.

One can check that the tensor fields $\phi^{(k)}$ do not depend on the trivializations and are well defined over $\widetilde{\mathbb{B}}^n$. Indeed, one has a sequence $\{\phi_J^{(k)}\}$ of deformation tensors over $\widetilde{\mathbb{B}}^n$ such that the series $\sum_{k=0}^{\infty}\phi_J^{(k)}$ converges uniformly on compact sets to ϕ_J.

These observations bring directly to the following corollary.

Corollary 3.6 *A manifold $(\mathbb{B}^n,J)$ of circular type in normal form, given by the blow down at 0 of $(\widetilde{\mathbb{B}}^n,J)$, is uniquely associated with a sequence of tensor fields $\phi_J^{(k)}$ in $(\mathscr{H}^{01})^*\otimes\mathscr{H}^{10}$, $0\leq k<\infty$, each of them (locally) of the form*

$$\phi_J^{(k)}([\mathrm{w}], \zeta\, \mathrm{w}) = \phi_J^k([\mathrm{w}])\zeta^k\ , \qquad \mathrm{w} \in S^{2n-1}\ ,\ \zeta \in \Delta\ ,$$

such that the series

$$\phi_J = \sum_{k\geq 0} \phi_J^{(k)} \tag{3.6}$$

converges uniformly on compacta and satisfies the following conditions:

i) $dd^c\tau_o(\phi_J(X), Y) + dd^c\tau_o(X, \phi_J(Y)) = 0$ *for anti-holomorphic* $X, Y \in H^{0,1}$*;*
ii) $\bar\partial_b \phi_J + \frac{1}{2}[\phi_J, \phi_J] = 0$*;*
iii)$dd^c\tau_o(\phi_J(X), \overline{\phi_J(X)}) < dd^c\tau_o(\bar X, X)$ *for any* $0 \neq X \in H^{0,1}$.

Conversely, any sequence of tensor fields $\phi_J^{(k)} \in (\mathscr{H}^{01})^* \otimes \mathscr{H}^{10}$*,* $0 \leq k < \infty$*, such that* (3.6)*converges uniformly on compacta and satisfies (i) - (iii), determines a manifold of circular type in normal form*

It is important to observe that the restriction $\widetilde{\phi}_J = \phi_J|_{S^{2n-1}(r)}$ of a deformation tensor ϕ_J to a sphere

$$S^{2n-1}(r) = \{\ ([\mathrm{v}], \mathrm{v})\ ,\ \|\mathrm{v}\| = r\ \} \subset \widetilde{\mathbb{B}}^n\ , \qquad 0 < r < 1$$

is the deformation tensor of the CR structure $(\mathscr{H}|_{S^{2n-1}(r)}, J)$, induced on $S^{2n-1}(r)$ by the complex structure J. Vice versa, any deformation tensor $\widetilde{\phi}_J$ of a CR structure of the form $(\mathscr{H}|_{S^{2n-1}(r)}, J)$ on a sphere $S^{2n-1}(r)$ can be written as a Fourier series

$$\widetilde{\phi}_J = \sum_{k\geq 0} \widetilde{\phi}^{(k)}\ , \tag{3.7}$$

whose terms are of the form

$$\widetilde{\phi}_J^{(k)}([\mathrm{w}], re^{i\vartheta}) = \widetilde{\phi}_J^k([\mathrm{w}]) r^k e^{ik\vartheta}\ , \qquad \mathrm{w} \in S^{2n-1}\ .$$

From this, one can see that all deformation tensors $\widetilde{\phi}_J$ of such CR structures are exactly the restrictions of the deformation tensors ϕ_J as in (3.6). If such deformation tensor ϕ_J satisfies conditions (i) - (iii) of Corollary 3.6, it uniquely determines a complex structure J which makes $(\widetilde{\mathbb{B}}^n, J)$ the blow up of a manifold of circular type in normal form.

Summing up all these observations, we see that there exists a natural bijection between the following two classes:

{Manifolds of circular type in normal formal
with the point 0 as distinguished center}

$$\Updownarrow$$

{Deformation tensors of CR-structures on $S^{2n-1}(r)$
satisfying suitable (explicit) conditions}

As we already mentioned, the Fourier developments of the tensors $\widetilde{\phi}_J$, which characterize the CR structures of the form $(\mathscr{H}|_{S^{2n-1}(r)}, J)$ on S^{2n-1} were first considered by Bland and Duchamp in [8, 9] in case of *small* deformations of the standard CR structure. They managed to prove that they can be always realized as the CR structures of boundaries of bounded domains in $\mathbb{C}^n$.

3.3 Parameterizations of normalizing maps

For a given manifold of circular type (M,J) (considered without a distinguished exhaustion τ), there are in general many distinct normalizing map

$$\Phi : M \longrightarrow \mathbb{B}^n .$$

The class $\mathscr{N}(M)$ of all such normalizing maps is an important biholomorphic invariant of the manifolds of circular type. Let us see in more details how such class $\mathscr{N}(M)$ can be studied.

Let us first try to understand the structure of the subclass $\mathscr{N}(M)_{x_o} \subset \mathscr{N}(M)$, consisting of the normalizing maps that send a fixed center x_o into the center 0 of its normal form $(\mathbb{B}^n, J')$. For this we need to introduced the notion of "special frames". Let $x_o \in M$ be the fixed center determined by a given Monge-Ampère exhaustion $\tau : M \longrightarrow \mathbb{R}_{\geq 0}$ and $\kappa = \kappa_{x_o} : T_{x_o}M = \mathbb{C}^n \longrightarrow \mathbb{R}_{\geq 0}$ the infinitesimal Kobayashi metric of M at x_o. Let also $I \subset T_{x_o}M$ be the Kobayashi indicatrix at x_o, that is

$$I = \{\ \mathrm{v} \in T_{x_o}M \simeq \mathbb{C}^n \ :\ \kappa_{x_o}(\mathrm{v}) < 1\ \} .$$

Recall that I is a circular domain of $T_{x_o}M \simeq \mathbb{C}^n$ and that κ coincides with the Minkowski functional of I. We call *special frame at x_o* any linear frame $(e_0, e_1, \ldots, e_{n-1})$ for $T_{x_o}M$ such that:

i) $e_0 \in \partial I$;
ii) $(e_1, \ldots, e_{n-1})$ is a collection of vectors in the tangent space

$$T_{e_0}(\partial I) \subset T_{e_0}(T_{x_o}M) = T_{x_o}M \simeq \mathbb{C}^n ,$$

that constitutes a linear frame for the holomorphic tangent space of ∂I at e_0, which is unitary w.r.t. the Levi form determined by the defining function $\rho = \kappa^2 - 1$ of I.

In [42], we proved the existence of a natural one-to-one correspondence between $\mathscr{N}(M)_{x_o}$ and the set P_{x_o} of all special frames at x_o. More precisely, given a fixed frame $u_o = (e^o_0, \ldots e^o_{n-1}) \in P_{x_o}$ and a normalizing map $\Phi_o \in \mathscr{N}(M)_{x_o}$, *the new basis of $T_{x_o}M$ defined by*

$$u^{\Phi} = \big(e_0, \ldots, e_{n-1}\big) \qquad \textit{with } e_i = (\Phi^{-1} \circ \Phi_o)_*(e^o_i\big) \ , \quad \Phi \in \mathscr{N}(M)_{x_o} \ ,$$

is also a special frame and the correspondence

$$\imath_{x_o} : \mathscr{N}_{x_o}(M) \longrightarrow P_{x_o} \ , \qquad \imath(\Phi) = u^{\phi}$$

is a bijection.

Therefore, for studying the whole class $\mathscr{N}(M)$ of normalizing maps, it is convenient to consider the collection of special frames

$$P(M) = \bigcup_{\substack{x \in M \text{ that} \\ \text{are centers w.r.t some } \tau}} P_x \ ,$$

which we call *pseudo-bundle of special frames*. We stress the fact that $P(M)$ is not expected to be a manifold – its geometric properties strongly depend on the geometry of the set of centers of M. However, it turns out that when M is a strictly linearly convex domain of $\mathbb{C}^n$, the pseudo-bundle $P(M)$ coincides with the unitary frame bundle of the complex Finsler metric, given by the infinitesimal Kobayashi metric κ (for definitions and properties of unitary frame bundles of complex Finsler manifolds, see e.g. [50]).

The previously defined correspondence between normalizing maps and special frames determines a natural bijection

$$\mathscr{N}(M) \stackrel{\sim}{\longrightarrow} P(M) \ .$$

Notice also that, if we identify M with one of its normal form $(\mathbb{B}^n, J)$, one can construct a diffeomorphism between the collection P_{x_o} of all special frames at a fixed center $x_o \in \mathbb{B}^n$ and the subgroup $\mathrm{U}_n = \mathrm{Aut}_0(\mathbb{B}^n, J_{\mathrm{st}})$ of the automorphisms of $(\mathbb{B}^n, J_{\mathrm{st}})$ fixing the origin. Such correspondence brings to an identification between $P(M)$ and a suitable subset of $\mathrm{Aut}(\mathbb{B}^n, J_{\mathrm{st}})$, which reveals to be a true diffeomorphism

$$\mathscr{N}(M) \stackrel{\sim}{\longrightarrow} \mathrm{Aut}(\mathbb{B}^n, J_{\mathrm{st}})$$

when M is a strictly linearly convex domain of $\mathbb{C}^n$.

3.4 Some geometrical interpretations and applications

Let $M = (\mathbb{B}^n, J, \tau_o)$ be a manifold of circular type in normal form, endowed with the standard exhaustion $\tau_o = \| \cdot \|^2$. Let also $\phi_J = \sum_{k=0}^{\infty} \phi_J^{(k)}$ be the corresponding deformation tensor and $I \subset T_0\mathbb{B}^n = \mathbb{C}^n$ the Kobayashi indicatrix at the center $x_o = 0$. Notice that, if we denote by $\mu = \kappa^2$ given by the square of the infinitesimal Kobayashi metric κ of $(\mathbb{B}^n, J)$ at 0, the pair (I, μ) is a domain of circular type in $T_0\mathbb{B}^n = \mathbb{C}^n$ – in fact, I is a circular domain and its Minkowski functional is κ!

One can prove the following.

Theorem 3.7

i) The 0*-th order component* $\phi_J^{(0)}$ *of* ϕ_J *coincides with the deformation tensor of the normal form of* (I,μ)*.*
ii) The difference $\phi_J - \phi_J^{(0)}$ *vanishes identically if and only if* M *is biholomorphic to the circular domain* I*.*

An application of this and all previous discussion is given by the following generalizations of results of Leung, Patrizio and P. M. Wong and for strictly convex domains and of Abate and Patrizio for Kähler-Finsler manifolds ([35, 1]). In the next statement, given a manifold of circular type (M,τ) with center x_o and a real number $0 < r < 1$, we use the notation $M_{x_o,r} = \{\, x \in M \ :\ \tau(x) < r \,\}$ for any $0 < r < 1$.

Theorem 3.8

1) A manifold of circular type (M,τ) *is biholomorphic to a circular domain if and only if the following condition holds:*

(⋆) there exists two distinct values $r_1, r_2 \in (0,1)$ *such that* M_{x_o,r_1} *is biholomorphic to* M_{x_o,r_2}*.*

2) A complex manifold (M,J) *is biholomorphic to the standard unit ball* $(\mathbb{B}^n, J_{\mathrm{st}})$ *if and only if it admits at least two distinct structures of manifold of circular type* (M,τ)*,* (M,τ')*, relative to two distinct centers* $x_o \neq x'_o$*, for which condition (⋆) holds.*

3.5 Remarks and Questions

Here are some open question, which we consider interesting and worth of investigations.

i) Find geometric interpretations of (possibly all) terms of the expansion in Fourier series $\phi_J = \sum_{k=0}^{\infty} \phi_J^{(k)}$ of the deformation tensor of a manifold of circular type in normal form.
ii) Using modular data (in practice, using possible expression for the deformation tensors in normal forms), construct explicit examples of manifolds of circular type with prescribed properties, e.g.,

– with exactly one center or with a discrete set of centers (if there are any);
– with an open set of centers:
– not embeddable in $\mathbb{C}^n$ (if there exist)

iii) Find conditions on the deformation tensor that characterize the domains of circular type, for which any point is a center.

We recall that P. M. Wong proved in [58] that any manifold of circular type admits non constant bounded holomorphic functions. In fact, such manifolds are hyperbolic and he proved that the Caratheodory metric of such manifolds is bounded below by a multiple of the Kobayashi metric. This stimulates further research towards the solution to the following basic question:

iv) Find conditions on modular data that characterize the manifolds of circular type biholomorphic to some strictly linearly convex domain or just to a bounded domain in $\mathbb{C}^n$.

4 The definition of "stationary disk" in the almost complex setting

4.1 *First definitions*

From now on, our discussion will focus on the wider class of *almost* complex manifolds and we will be mainly concerned with generalizations of previous results in this larger context.

In what follows, M is always a $2n$-dimensional real manifold with an almost complex structure J, which is a tensor field of type $(1,1)$ that gives a linear map at any $x \in M$

$$J_x : T_xM \longrightarrow T_xM \qquad \text{such that } \; J_x^2 = -\operatorname{Id}_{T_xM} \ .$$

We recall that an almost complex structure J is called *integrable* if the associated Nijenhuis tensor N_J vanishes identically. The definition of N_J is the following: It is the tensor field of type $(1,2)$ defined by the relation

$$N(X,Y) = \frac{1}{4}\left([X,Y] - [JX,JY] + J[JX,Y] + J[X,JY]\right)$$

for any pair of vector fields X,Y of M.

By the celebrated Newlander-Nirenberg theorem, an almost complex structure J is integrable if and only if M admits a structure of complex manifold, i.e., if and only if there exists an atlas of complex charts for M

$$\xi = (z^1 = x^1 + iy^1, \dots, z^n = x^n + iy^n) : \mathscr{U} \subset M \longrightarrow \mathbb{C}^n \ ,$$

such that

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}\ , \qquad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}$$

and the changes of coordinates $\xi\circ\eta^{-1}$, $\eta\circ\xi^{-1}$ between any two overlapping charts of the atlas are holomorphic. In the following, the charts of such atlas will be called *systems of holomorphic coordinates*.

Given a pair of almost complex manifold (M,J), (M',J'), a map $f : M \longrightarrow M'$ is called (J,J')-*holomorphic* (or, simply, *holomorphic*) if

$$\overline{\partial}_{J,J'} f(\mathrm{v}) = 0 \qquad \text{for any } \mathrm{v} \in TM\ ,$$

where $\overline{\partial}_{J,J'} f$ is the map

$$\overline{\partial}_{J,J'} f : TM \longrightarrow TM'\ , \qquad \overline{\partial}_{J,J'} f(\mathrm{v}) = f_*(J\mathrm{v}) - J'(f_*(\mathrm{v}))\ . \tag{4.1}$$

Notice that $\overline{\partial}_{J,J'}$ is a natural generalization of the usual $\overline{\partial}$-operator. In fact, when (M,J) and (M',J') are complex manifolds (i.e., when J, J' are both integrable) and

$$\xi = \left(x^i = \frac{z^i+\overline{z^i}}{2}, y^i = \frac{z^i-\overline{z^i}}{2i}\right)\ , \qquad \xi' = \left(x'^i = \frac{z'^i+\overline{z'^i}}{2}, y'^i = \frac{z'^i-\overline{z'^i}}{2i}\right)$$

are systems of holomorphic coordinates for M and M', respectively, the expression of an arbitrary smooth real map $f : M \longrightarrow M'$ is of the form

$$f(z^i,\overline{z^j}) = (f^m(z^i,\overline{z^j}), f^{\overline{m}}(z^i,\overline{z^n}))\ ,$$

where $f^m(z^i,\overline{z^j})$ and $f^{\overline{m}}(z^i,\overline{z^j})$ denote the values of $f(z^i,\overline{z^j})$ in the complex coordinates z'^m and $\overline{z'^m}$ of M'. In such coordinates the ($\mathbb{C}$-linear extended) map

$$\overline{\partial}_{J,J'} f : T^{\mathbb{C}}M \longrightarrow T^{\mathbb{C}}M$$

is

$$\begin{aligned}\overline{\partial}_{J,J'} f\left(\frac{\partial}{\partial\overline{z}^i}\right) &= -i\frac{\partial f^j}{\partial\overline{z^i}}\frac{\partial}{\partial z'^j} - i\frac{\partial f^{\overline{j}}}{\partial\overline{z}^i}\frac{\partial}{\partial\overline{z'^j}} - i\frac{\partial f^j}{\partial\overline{z^i}}\frac{\partial}{\partial z'^j} + i\frac{\partial f^j}{\partial\overline{z^i}}\frac{\partial}{\partial\overline{z'^j}} = \\ &= -i2\frac{\partial f^j}{\partial\overline{z^i}}\frac{\partial}{\partial z'^j}\ ,\end{aligned}$$

showing that $\overline{\partial}_{J,J'} f$ vanishes identically if and only if the f^ℓ's are holomorphic in the usual sense.

In the following, when $(M,J) \subset (\mathbb{C}^n, J_{\mathrm{st}})$, we will often use the simplified notation $\overline{\partial}_{J'} = \overline{\partial}_{J_{\mathrm{st}},J'}$. Given an almost complex manifold (M,J), a map

$$f : \Delta = \{\ |\zeta| < 1\ \} \longrightarrow (M,J)$$

is called *J-holomorphic disk* if it is (J_{st}, J)-holomorphic or, equivalently, if $\overline{\partial}_J f = 0$. A simple argument shows that, for an arbitrary $\mathscr{C}^1$-map $f : \Delta \longrightarrow (M,J)$, the J-holomorphicity condition $\overline{\partial}_J f = 0$ is equivalent to the differential equation

$$\partial_J f\left(\left.\frac{\partial}{\partial x}\right|_{\zeta}\right) = 0 \qquad \text{for any } \zeta = x + iy \in \Delta$$

(see e.g. [27, 51]).

4.2 Canonical lifts of almost complex structures

When (M,J) is a (integrable) complex manifold, it is easy to define a corresponding pair of natural complex structures on the tangent bundle

$$\widehat{\pi} : TM \longrightarrow M$$

and on the cotangent bundle

$$\widetilde{\pi} : T^*M \longrightarrow M\ .$$

In fact, any system of holomorphic coordinates $\xi = (x^i)$ can be used to locally identify M, TM and T^*M with open subsets of $\mathbb{C}^n$, $T\mathbb{C}^n = \mathbb{C}^{2n}$ and $T^*\mathbb{C}^n = \mathbb{C}^{2n}$, respectively. These identifications determine integrable almost complex structures $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ on TM and T^*M, which turn out to be independent on the choice of the considered system of holomorphic coordinates. They are therefore *naturally and globally defined complex structures on TM and T^*M, respectively*.

When (M,J) is a (non-integrable) almost complex manifold, the notion of "system of holomorphic coordinates" is meaningless and the above construction does not apply. However, it is still possible to define a pair of almost complex structures on TM and T^*M, which depend in a canonical way on the almost complex J of M. Such almost complex structures were introduced by Yano and Ishihara in the '70's and are defined as follows (see [59]). Given a system of coordinates

$$\xi = (x^1, \ldots, x^{2n}) : \mathscr{U} \subset M \longrightarrow \mathbb{R}^n\ ,$$

let us denote by

$$\widehat{\xi} = (x^1, \ldots, x^{2n}, q^1, \ldots, q^{2n}) : \widehat{\pi}^{-1}(\mathscr{U}) \subset TM \longrightarrow \mathbb{R}^{4n}\ ,$$

$$\widetilde{\xi} = (x^1, \ldots, x^{2n}, p_1, \ldots, p_{2n}) : \widetilde{\pi}^{-1}(\mathscr{U}) \subset T^*M \longrightarrow \mathbb{R}^{4n}\ ,$$

the associated coordinates on $TM|_{\mathscr{U}}$ and $T^*M|_{\mathscr{U}}$, determined by the components q^i of the vectors $\mathrm{v} = q^i \frac{\partial}{\partial x^i}$ in the basis $\left(\frac{\partial}{\partial x^i}\right)$ and by the components p_j of the covectors $\alpha = p_j dx^j$ in the basis (dx^i). Let us also denote by $J^i_j = J^i_j(x)$ the components of the almost complex structure $J = J^i_j \frac{\partial}{\partial x^i} \otimes dx^j$.

The *canonical lifts of J on TM and T^*M* are the almost complex structures $\mathbb{J}$ on TM and $\widetilde{\mathbb{J}}$ on T^*M, defined by

$$\mathbb{J} = J^a_i \frac{\partial}{\partial x^a} \otimes dx^i + J^a_i \frac{\partial}{\partial q^a} \otimes dq^i + q^b J^a_{i,b} \frac{\partial}{\partial q^a} \otimes dx^i \ , \tag{4.2}$$

$$\begin{aligned} \widetilde{\mathbb{J}} = J^a_i \frac{\partial}{\partial x^a} \otimes dx^i + J^a_i \frac{\partial}{\partial p_i} \otimes dp_a + \\ + \frac{1}{2} p_a \left(-J^a_{i,j} + J^a_{j,i} + J^a_\ell \left(J^\ell_{i,m} J^m_j - J^\ell_{j,m} J^m_i \right) \right) \frac{\partial}{\partial p_j} \otimes dx^i \ . \end{aligned} \tag{4.3}$$

These tensor fields can be checked to be independent on the chart (x^i) and:

i) the standard projections $\widehat{\pi} : T^*M \longrightarrow M$, $\widetilde{\pi} : T^*M \longrightarrow M$ are $(\mathbb{J}, J)$-holomorphic and $(\widetilde{\mathbb{J}}, J)$-holomorphic, respectively;
ii) given a (J, J')-biholomorphism $f : (M, J) \longrightarrow (N, J')$ between almost complex manifolds, the tangent and cotangent maps

$$f_* : TM \longrightarrow TN \qquad \text{and} \qquad f^* : T^*N \longrightarrow T^*M$$

are $(\mathbb{J}, \mathbb{J}')$- and $(\widetilde{\mathbb{J}}', \widetilde{\mathbb{J}})$-holomorphic, respectively;
iii) when J is integrable, $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ coincide with above described integrable complex structures of TM and T^*M, respectively.

In order to better understand the property (iii) and see the precise relation between the almost complex structures $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ with their analogues of the integrable case, it is convenient to rewrite them in (non-holomorphic) complex coordinates, i.e., in complex coordinates of the form

$$(z^A) = (z^a = x^a + i x^{a+n}, z^{\overline{a}} = \overline{z^a} = x^a - i x^{a+n}) \ .$$

If $(q^A) = (q^a, \overline{q}^a)$ and $(p_A) = (p_a, p_{\overline{a}} \stackrel{\text{def}}{=} \overline{p_a})$ are the complex components of *real* vector fields and *real* 1-forms

$$X = q^a \frac{\partial}{\partial z^a} + \overline{q^a} \frac{\partial}{\partial \overline{z}^a} \in TM \ , \qquad \omega = p_a dz^a + \overline{p}_a dz^{\overline{a}} \in T^*M \ ,$$

the canonical lifts $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ can re-written in the following form:

$$\mathbb{J} = J^A_B \left(\frac{\partial}{\partial z^A} \otimes dz^B + \frac{\partial}{\partial q^A} \otimes dq^B \right) + q^C J^A_{B,C} \frac{\partial}{\partial q^A} \otimes dx^B \ ,$$

$$\widetilde{\mathbb{J}} = J^B_A \left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B \right) +$$
$$+ \frac{1}{2} p_C \left(-J^C_{A,B} + J^C_{B,A} + J^C_L \left(J^L_{A,M} J^M_B - J^L_{B,M} J^M_A \right) \right) \frac{\partial}{\partial p_B} \otimes dz^A \ ,$$

where the J^A_B's are the components of J w.r.t. the complex vector fields $\left(\frac{\partial}{\partial z^A}\right)$. When J is integrable and $(z^1, \ldots, z^n)$ are holomorphic coordinates, the J^A_B's are constant and equal to the entries of the matrix

$$\left(J^A_B \right) = \begin{pmatrix} i\delta^a_b & 0 \\ 0 & -i\delta^{\overline{a}}_{\overline{b}} \end{pmatrix} \ .$$

and $\mathbb{J}$ and $\widetilde{\mathbb{J}}$ assume the familiar expressions

$$\mathbb{J} = J^A_B \left(\frac{\partial}{\partial z^A} \otimes dz^A + \frac{\partial}{\partial q^A} \otimes dq^B \right) \ , \ \widetilde{\mathbb{J}} = J^B_A \left(\frac{\partial}{\partial z^B} \otimes dz^A + \frac{\partial}{\partial p_A} \otimes dp_B \right) \ .$$

4.3 Strong pseudoconvexity in the almost complex setting

The classical notions of “CR structure”, “Levi form”, “(strong) pseudoconvexity” admit direct and simple generalizations in the context of almost complex manifolds. Let us recall them.

Let (M, J) be an almost complex manifold and $\Gamma \subset M$ a connected (smooth) hypersurface. The *(induced) CR structure of* Γ is the pair $(\mathscr{D}, J^{\mathscr{D}})$ formed by

a) the distribution $\mathscr{D} \subset TM$ defined by

$$\mathscr{D} = \bigcup_{x \in M} \mathscr{D}_x \ , \qquad \mathscr{D}_x = \{ \, \mathrm{v} \in T_x\Gamma \ : \ Jv \in T_x\Gamma \, \} \ ;$$

b) the family $J^{\mathscr{D}}$ of complex structures

$$J^{\mathscr{D}}_x : \mathscr{D}_x \longrightarrow \mathscr{D}_x \ , \qquad J^{\mathscr{D}}_x(\mathrm{v}) = J_x \mathrm{v} \ .$$

A 1-form ϑ in $T^*\Gamma$ is called *defining form for* $\mathscr{D}$ if for any $x \in \Gamma$

$$\ker \vartheta_x = \mathscr{D}_x \ .$$

Notice that, for any pair of defining forms ϑ, ϑ' for $\mathscr{D}$, there exists a nowhere vanishing, smooth real function λ such that

$$\vartheta' = \lambda \cdot \vartheta \ . \tag{4.4}$$

In the following, we will assume that Γ is oriented, i.e., endowed with a fixed choice of a nowhere vanishing vector field $\xi \in T\Gamma \setminus \mathscr{D}$. Clearly, $N_x = J\xi_x$ is transversal to $T_x\Gamma$ at all points $x \in \Gamma$ and to be oriented in the previous sense coincide with the usual definition. If $\Gamma = \partial D$ is the boundary of a relatively compact domain $D \subset M$, we will always assume that the orientating field ξ is such that the vector $N_x = J\xi_x$ is pointing outwards D for any $x \in \Gamma$. A defining form ϑ such that $\vartheta(\xi) > 0$ (resp. < 0) will be called *positive* (resp. *negative*).

Given a fixed positive defining form ϑ, the *Levi form of Γ at x* is the quadratic form

$$\mathscr{L}_x : \mathscr{D}_x \longrightarrow \mathbb{R}\ , \qquad \mathscr{L}_x(\mathrm{v}) = d\vartheta_x(\mathrm{v}, J\mathrm{v}) = \vartheta_x([X^{(\mathrm{v})}, JX^{(\mathrm{v})}])\ , \tag{4.5}$$

where $X^{(\mathrm{v})}$ is any smooth vector field with values in $\mathscr{D}$ such that $X^{(\mathrm{v})}_x = \mathrm{v}$. From the last expression in (4.5), it follows immediately that if ϑ is replaced by another positive defining form, the corresponding Levi form changes only by a positive factor.

Definition 4.1 An oriented smooth hypersurface $\Gamma \subset M$ is called *strongly pseudoconvex* if $\mathscr{L}_x > 0$ for any $x \in \Gamma$. A smooth, relatively compact domain $D \subset M$ is called *strongly pseudoconvex* if ∂D is strongly pseudoconvex.

Many properties of classical strongly pseudoconvex domains generalize to the case of strongly pseudoconvex domains in almost manifolds. For instance, it is known that $D \subset M$ *is strongly pseudoconvex if and only if it admits a strictly J-plurisubharmonic defining function* (for the definition of J-plurisubharmonicity, see later). For this and other basic properties of almost complex domains, we refer to the survey [16].

4.4 Stationary disks

Let us now introduce the notion of stationary disks of almost complex strongly pseudoconvex domains. The original definition of stationary disk is due to Lempert [31] (see also [49]) and extended by Tumanov [54] to more general settings for submanifolds in (integrable) complex manifolds. Tumanov’s definition extends directly to the almost complex environment (see [15, 16, 22]). As before, (M, J) is an almost complex manifold and $\Gamma = \partial D \subset M$ is the oriented, smooth hypersurface, which is the boundary of a relatively compact domain $D \subset M$.

Let us recall that the *conormal bundle of Γ* is the collection $\mathscr{N}$ of 1-forms at the points of Γ defined by

$$\mathscr{N} = \{\ \alpha \in T^*_x M\ :\ x \in \Gamma \text{ and } T_x\Gamma \subset \ker\alpha\ \} \subset T^*M|_\Gamma\ .$$

We denote $\mathscr{N}_* = \mathscr{N} \setminus \{\text{zero section}\}$.

Definition 4.2 Given $\alpha \geq 1$, $\varepsilon > 0$, a map $f : \overline{\Delta} \to M$ from the closed unit disk $\overline{\Delta} \subset \mathbb{C}$ into M is called $\mathscr{C}^{\alpha,\varepsilon}$*-stationary disk of* D if

i) $f|_\Delta$ is a J-holomorphic embedding and $f(\partial\Delta) \subset \partial D$;
ii) there exists a $\widetilde{\mathbb{J}}$-holomorphic maps $\widetilde{f} : \overline{\Delta} \to T^*M$ with $\pi \circ \widetilde{f} = f$ and $\widetilde{\xi} \circ \widetilde{f}$ in $\mathscr{C}^{\alpha,\varepsilon}(\overline{\Delta}, \mathbb{C}^{2n})$ for some system of complex coordinates $\widetilde{\xi} = (z^i, w_j)$, such that

$$\zeta^{-1} \cdot \widetilde{f}(\zeta) \in \mathscr{N}_* \quad \text{for any } \zeta \in \partial\Delta \ . \tag{4.6}$$

If f is a stationary disk, the maps $\widetilde{f}$ that satisfy (ii) are called *stationary lifts of f*

In (4.6), the product " $\cdot$ " denotes the $\mathbb{C}$-action on T^*M defined by

$$\zeta \cdot \alpha = \mathrm{Re}(\zeta)\alpha - \mathrm{Im}(\zeta)J^*\alpha \quad \text{for any } \alpha \in T^*M,\ \ \zeta \in \mathbb{C} \ . \tag{4.7}$$

We point out that, as a consequence of the maximum principle for subharmonic functions and of the fact that D is strongly pseudoconvex, for any disk $f : \overline{\Delta} \longrightarrow M$ satisfying (i), one has that $f(\overline{\Delta}) \subset \overline{D}$ and $f(\zeta) \in \partial D$ if and only if $\zeta \in \partial\Delta$.

Remark 4.3 When J is an integrable complex structure, condition (ii) implies that the restriction along $f(\partial\Delta)$ of the CR distribution of ∂D extends to a $\widetilde{\mathbb{J}}$-holomorphic bundle over Δ ($\simeq f(\Delta)$), this being a characterizing property of the usual stationary disks of the domains of $\mathbb{C}^n$ ([31]). This is one of the reasons why the previous definition can be considered as the natural generalization of the concept of stationary disk in the almost complex setting.

5 Almost complex domains of circular type

5.1 Looking for the stationary disks of almost complex domain

In this and the next sections, D is a smooth, relatively compact, strongly pseudoconvex domain in an almost complex manifold (M, J) with boundary $\Gamma = \partial D$ and

$$\mathscr{N}_* = \mathscr{N} \setminus \{\text{zero section}\}\ , \ \text{ where } \mathscr{N} \subset T^*M|_{\partial D} \ \text{ conormal bundle}\ .$$

We will also assume that:

– $\overline{D} \subset M$ is contained in a globally coordinatizable open subset $\mathscr{U} \subset M$ or, equivalently, D is a domain of $M = \mathbb{R}^{2n} \simeq \mathbb{C}^n$ equipped with a non-standard complex structure J;
– D has a smooth global defining function $\rho : \mathscr{U} \subset M \longrightarrow \mathbb{R}$, i.e.,

$$D = \{\ x \in M\ :\ \rho(x) < 0\ \}$$

with $d\rho_x \neq 0$ for any $x \in \Gamma = \partial D$.

Let us study the differential problem that characterizes the lifts $\widetilde{f} : \overline{\Delta} \to T^*M$ of stationary disks of D. Consider the map

$$\widetilde{\rho} : \mathbb{R}_* \times T^*M|_{\mathscr{U}} \longrightarrow \mathbb{R} \times T^*M|_{\mathscr{U}} \ , \ \ \widetilde{\rho}(t,\alpha) \stackrel{\text{def}}{=} (\rho(\widetilde{\pi}(\alpha)), \alpha - t \cdot d\rho_{\widetilde{\pi}(\alpha)}) \ . \tag{5.1}$$

Notice that $\mathscr{N}_*$ is a $2n$-dimensional submanifold of T^*M and that it can be identified with the level set

$$\{(t,\alpha) : \ t \neq 0 \ , \ \widetilde{\rho}(t,\alpha) = (0_{\mathbb{R}}, 0_{T^*_{\widetilde{\pi}(\alpha)}M})\} \subset \mathbb{R}_* \times T^*M|_{\mathscr{U}} \ ,$$

which is a $2n$-dimensional submanifold of $\mathbb{R}_* \times T^*M$. Therefore, using a system of coordinates $\widetilde{\xi} = (x^i, p_j)$ on $T^*M|_{\mathscr{U}}$, associated with coordinates $\xi = (x^i)$, we may identify $\mathbb{R}_* \times T^*M|_{\mathscr{U}}$ with an open subset $\mathscr{V} \subset \mathbb{R}^{4n+1}$ and $\mathscr{N}_*$ with the level set in $\mathscr{V}$ defined by

$$\mathscr{N}_* \simeq \{ \ (t,\alpha) \in \mathscr{V} \ : \ \widetilde{\rho}^i(t,\alpha) = 0 \ , \ \ 1 \leq i \leq 2n+1 \} \ .$$

By a direct check of the rank of the Jacobian, one can see that $\widetilde{\rho} = (\widetilde{\rho}^1, \dots, \widetilde{\rho}^{2n+1})$ *is a smooth defining function for* $\mathscr{N}_*$.

We now consider the map $r : \mathbb{C} \times \mathscr{V} \subset \mathbb{C} \times \mathbb{R}^{4n+1} \longrightarrow \mathbb{R}^{2n+1}$, defined by

$$r(\zeta,t,\alpha) \stackrel{\text{def}}{=} \left(\widetilde{\rho}^1(t, \zeta^{-1} \cdot \alpha), \dots, \widetilde{\rho}^n(t, \zeta^{-1} \cdot \alpha)\right) \ . \tag{5.2}$$

By the above identifications, we have that a disk $f : \overline{\Delta} \to \overline{D} \subset \mathbb{R}^{2n}$ is stationary if and only if there exists $\widetilde{f} \in \mathscr{C}^{\alpha,\varepsilon}(\bar{\Delta}; \mathbb{C}^{2n})$ and $\lambda \in \mathscr{C}^{\varepsilon}(\partial\Delta; \mathbb{R})$ such that

$$\begin{cases} \overline{\partial}_{\mathbb{J}} \widetilde{f}(\zeta) & = 0, \quad \zeta \in \Delta \ , \\ \\ r(\zeta, \lambda(\zeta), \widetilde{f}(\zeta)) & = 0, \quad \zeta \in \partial\Delta \ , \end{cases} \tag{5.3}$$

where $\overline{\partial}_{\mathbb{J}} = \overline{\partial}_{J_{\mathrm{st}},\mathbb{J}} : \mathscr{C}^{\alpha}(\overline{\Delta}; \mathbb{C}^{2n}) \longrightarrow \mathscr{C}^{\alpha-1}(\overline{\Delta}; \mathbb{C}^{2n})$ is the operator (4.1). Problem (5.3) belongs to the class usually called *of generalized Riemann-Hilbert problems*, for which there exists a well developed theory (see e.g. [36, 55]).

In order to study the solution space of (5.3) and its stability w.r.t. small deformations of data, one has to find explicit coordinate expressions for the operators, which determines this problem. For this, let us fix an almost complex structure $J = J_o$, a point $x_o \in D(\subset \mathbb{R}^{2n})$ and a vector $\mathrm{v}_o \in T_{x_o}D \simeq \mathbb{R}^{2n}$. Denote by $\mathscr{R}_1, \dots, \mathscr{R}_5$ the operators

$$(\widetilde{f}, \lambda, \mu) \in \mathscr{C}^{\alpha,\varepsilon}(\overline{\Delta}; \mathbb{C}^{2n}) \times \mathscr{C}^{\varepsilon}(\partial\Delta; \mathbb{R}) \times \mathbb{R}_* \ ,$$

which correspond to the conditions of (5.3) plus some additional conditions, which are convenient to introduce in order to fully parameterize the solution space of the problem:

$$\begin{aligned}
&\mathscr{R}_1(\widetilde{f},\lambda,\mu) = \overline{\partial}_{\mathbb{J}_o}\widetilde{f} && (\widetilde{\mathbb{J}}\textit{-holomorphicity of } \widetilde{f})\ ,\\
&\mathscr{R}_2(\widetilde{f},\lambda,\mu) = r(\zeta,\lambda(\zeta),\widetilde{f}(\zeta)) && (\textit{boundary data for } \widetilde{f})\ ,\\
&\mathscr{R}_3(\widetilde{f},\lambda,\mu) = \widetilde{\pi}(\widetilde{f})|_{\zeta=0} - x_o && (\textit{center of } f = \widetilde{\pi}\circ\widetilde{f})\ ,\\
&\mathscr{R}_4(\widetilde{f},\lambda,\mu) = \widetilde{\pi}(\widetilde{f})_*\left(\left.\tfrac{\partial}{\partial x}\right|_{\zeta=0}\right) - \mu\,\mathrm{v}_o && (\textit{tangent vector at the center})\ ,\\
&\mathscr{R}_5(\widetilde{f},\lambda,\mu) = \widetilde{f}\left(\widetilde{\pi}(\widetilde{f})_*\left(\left.\tfrac{\partial}{\partial x}\right|_1\right)\right) - 1 && (\textit{normalizing condition on } \widetilde{f})\ .
\end{aligned} \tag{5.4}$$

Using Hopf's Lemma, one can check that, for any stationary disk, there exists exactly one stationary lift satisfying the condition

$$\widetilde{f}\left(\widetilde{\pi}(\widetilde{f})_*\left(\left.\frac{\partial}{\partial x}\right|_1\right)\right) = 1\ .$$

Therefore, if we denote by $\mathscr{R}_{(J_o,x_o,v_o)} = (\mathscr{R}_1,\dots,\mathscr{R}_5)$ the operator

$$\mathscr{R}_{(J_o,x_o,v_o)} = (\mathscr{R}_1,\dots,\mathscr{R}_5) : \mathscr{C}^{\alpha,\varepsilon}(\overline{\Delta};\mathbb{C}^{2n})\times\mathscr{C}^{\varepsilon}(\partial\Delta;\mathbb{R})\times\mathbb{R}_* \longrightarrow$$

$$\longrightarrow \mathscr{C}^{\alpha-1,\varepsilon}(\overline{\Delta};\mathbb{C}^{2n})\times\mathscr{C}^{\varepsilon}(\partial\Delta;\mathbb{R}^{2n+1})\times\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{R}\ ,$$

we see that there is a one-to-one correspondence between the stationary disks $f : \overline{\Delta} \longrightarrow \overline{D}$ with $f(0) = x_o$ and $f_*\left(\left.\frac{\partial}{\partial x}\right|_{x_o}\right)$ and the solutions to the problem

$$\mathscr{R}_{(J_o,x_o,v_o)}(\widetilde{f},\lambda,\mu) = 0\ . \tag{5.5}$$

The following is a well-known fact of Lempert's theory of stationary disks ([31]): *If $D \subset \mathbb{C}^n$ is a strictly (linearly) convex, smoothly bounded domain, endowed with the standard complex structure $J_o = J_{\mathrm{st}}$, for any $x_o \in D$ and $\mathrm{v}_o \in T_{x_o}D$, the problem* (5.5) *has a unique solution smoothly depending on data x_o and v_o.*

Now, a similar existence and uniqueness result and a smooth dependence on the data for the stationary disks of almost complex domains (D,J) can be proved whenever J is small deformations of J_{st} in the following sense.

Consider a solution $(\widetilde{f}_o,\lambda_o,\mu_o)$ of (5.5) and denote by

$$\mathfrak{R}_{(J_o,x_o,\mathrm{v}_o;\widetilde{f}_o,\lambda_o,\mu_o)} \overset{\mathrm{def}}{=} \dot{\mathscr{R}}_{(J_o,x_o,\mathrm{v}_o)}|_{(\widetilde{f}_o,\lambda_o,\mu_o)}$$

the linearized operator at $(\widetilde{f}_o,\lambda_o,\mu_o)$ determined by $\mathscr{R}_{(J_o,x_o,\mathrm{v}_o)}$. By the Implicit Function Theorem (see e.g. [29]), when the linear operator $\mathfrak{R} = \mathfrak{R}_{(J_o,x_o,\mathrm{v}_o;\widetilde{f}_o,\lambda_o,\mu_o)}$ is invertible, there exists a solution to the problem $\mathscr{R}_{(J_t,x_t,\mathrm{v}_t)}(\widetilde{f},\lambda,\mu) = 0$ for any smooth deformation (J_t,x_t,v_t) of (J_o,x_o,v_o) for a sufficiently small t. In this case

$\dim_{\mathbb{R}} \ker \mathfrak{R}_{(J_o,x_o,\mathrm{v}_o;\widetilde{f}_o,\lambda_o,\mu_o)}$ is equal to the dimension of the solutions space. This fact motivates the following definition.

Definition 5.1 Let $f_o : \overline{\Delta} \to \overline{D}$ be a stationary disk of (D,J_o) with $x_o = f(0)$ and $\mathrm{v}_o = f_* \left(\frac{\partial}{\partial x}\Big|_{\zeta=0} \right)$. We say that ∂D *is a good boundary for* (J_o, f_o) if there exists a lift $\widetilde{f_o}$ of f_o and a function λ_o such that $(\widetilde{f}_o, \lambda_o, 1)$ is a solution to (5.5) and the linearized operator $\mathfrak{R} = \mathfrak{R}_{(J_o,x_o,\mathrm{v}_o;\widetilde{f}_o,\lambda_o,1)}$ is invertible.

The Implicit Function Theorem and previous remarks bring immediately to the next proposition. In the statement, we denote by $g = g_{ij}dx^i \otimes dx^j$ a fixed Riemannian metric on a neighborhood of $\overline{D}$ and by $g^* = g_{ij}dx^i \otimes dx^j + g^{ij}dp_i \otimes dp_j$ the corresponding Riemannian metric on T^*M. We also set

$$\|J - J'\|^{(1)}_{\overline{D}} \overset{\text{def}}{=} \sup_{x \in \overline{D}, v \in T(T^*_x M)} \frac{\|\mathbb{J}(\mathrm{v}) - \mathbb{J}'(\mathrm{v})\|_{g^*}}{\|\,\mathrm{v}\,\|_g} \;, \tag{5.6}$$

where $\|\cdot\|_g$ and $\|\cdot\|_{g^*}$ are the norm functions determined by g and g^*. The topology determined by the norm $\|\cdot\|^{(1)}_{\overline{D}}$ is clearly independent on the choice of g.

Proposition 5.2 *Let $f_o : \overline{\Delta} \to \overline{D}$ be a stationary disk of $D \subset (M, J_o)$ with $x_o = f_o(0)$ and $\mathrm{v}_o = f_{o*}\left(\frac{\partial}{\partial x}\Big|_{\zeta=0} \right)$. Assume also that ∂D is a good boundary for (J_o, f_o).*

Then, there exists neighborhoods $\mathscr{V} \subset D$, $\mathscr{W} \subset TD$ of x_o and v_o, with $\widetilde{\pi}(\mathscr{W}) = \mathscr{V} \subset D$, and $\varepsilon > 0$ such that, for any

$$x \in \mathscr{V} \;, \qquad \mathrm{v} \in \mathscr{W} \;, \qquad \|J - J_o\|^{(1)}_{\overline{D}} < \varepsilon \;,$$

there exists a unique stationary disk f with

$$f(0) = x \quad \text{and} \quad f_*\left(\frac{\partial}{\partial x}\Big|_{\zeta=0} \right) = \mu\, \mathrm{v} \quad \text{for some} \;\; \mu \neq 0 \;. \tag{5.7}$$

The disk f depends smoothly on x, v and J.

The previous result reduces the problem of finding domains with well-behaved families of stationary disks to the query for almost complex domains (D,J) with stationary disks and good boundaries for such disks – let us informally call this kind of domains “good”. By these remarks, we can determined useful and efficient results if we are able to determine conditions that ensure that a domain is “good”. Such conditions do exist and we will shortly discuss them.

For the moment, let us see what one can do if he has to deal with a “good” almost complex domain. As usual, let $D \subset (M,J)$ be a smooth, relatively compact, strongly pseudoconvex domain in almost complex manifold. For a fixed $x_o \in D$, let $\pi : \widetilde{M} \longrightarrow M$ be the *blow up of M at x_o*. Here some care is needed: Keep in mind that the definition of "blow up at a point" is usually defined just for *complex* manifolds. Nonetheless there exists a generalization that makes sense also in case of almost complex domains and this is the notion we refer to (for details, see e.g. [44]).

Now, if $f : \overline{\Delta} \longrightarrow \overline{D}$ is a stationary disk with $f(0) = x_o$ and $f_*\left(\left.\frac{\partial}{\partial x}\right|_0\right) = w$, we may define a J-holomorphic lift of f with image in the closure $\overline{\widetilde{D}}$ in $\widetilde{M}$ of the blow up $\widetilde{D}$ at x_o:

$$\widehat{f} : \overline{\Delta} \longrightarrow \overline{\widetilde{D}} \subset \widehat{M}\ , \qquad \widehat{f}(\zeta) = \begin{cases} (f(\zeta), [f(\zeta)]) & \text{if } \zeta \neq 0\ , \\ (x_o, [w]) & \text{if } \zeta = 0\ . \end{cases}$$

This allows to consider the next definition.

Definition 5.3 Let $x_o \in D$ and $\overline{\widetilde{D}}$ as above and denote by $\mathscr{F}^{(x_o)}$ the family of all stationary disks of D with $f(0) = x_o$. We say that $\mathscr{F}^{(x_o)}$ is a *foliation of circular type of the pointed domain* (D, x_o) if the following conditions are satisfied:

i) for any $\mathrm{v} \in T_{x_o}D$, there exists a unique disk $f^{(\mathrm{v})} \in \mathscr{F}^{(x_o)}$ such that $f^{(\mathrm{v})}_*\left(\left.\frac{\partial}{\partial x}\right|_0\right) = \mu \cdot \mathrm{v}$ for some $0 \neq \mu \in \mathbb{R}$;
ii) under a fixed identification $(T_{x_o}D, J_{x_o}) \simeq (\mathbb{C}^n, J_{\mathrm{st}})$, the map

$$E : \widetilde{\mathbb{B}}^n \subset \widetilde{\mathbb{C}}^n \longrightarrow \widetilde{D}\ , \qquad E(\mathrm{v}, [\mathrm{v}]) = \widetilde{f^{(\mathrm{v})}}(|\,\mathrm{v}\,|)\ , \tag{5.8}$$

between the blow ups of $\mathbb{B}^n \subset \mathbb{C}^n$ and D at 0 and x_o, respectively, is smooth and extends smoothly up to the boundary, determining a diffeomorphism $E|_{\partial\mathbb{B}^n} : \partial\mathbb{B}^n \longrightarrow \partial D$.

If $\mathscr{F}^{(x_o)}$ is a foliation of circular type, we say that x_o is the *center of the foliation* and that D is a *domain of circular type with center* x_o.

Proposition 5.2 brings to the following stability result for foliation of circular type of "good" domains.

Proposition 5.4 *Let D be of circular type w.r.t. to an almost complex structure J_o and with center x_o, such that ∂D is a good boundary for (J_o, f) for any $f \in \mathscr{F}^{(x_o)}$.*

Then there exists $\varepsilon > 0$ and an open neighborhood $\mathscr{U} \subset D$ of x_o such that for all almost complex structures J (defined on a neighborhood of $\overline{D}$) with $\|J - J_o\|^{(1)}_{\overline{D}} < \varepsilon$, D is a domain of circular type w.r.t. J with center $x \in \mathscr{U}$ (i.e., for any such J and x, the corresponding collection of stationary disks $\mathscr{F}^{(x)}$ is a foliation of circular type).

Let us now come to the main results on existence of foliations by stationary disks.

Theorem 5.5 ([43]) *Let $D \subset M$ be a bounded, relatively compact, strongly pseudoconvex domain with smooth boundary in an almost complex manifold (M, J_o). If there exists a diffeomorphism*

$$\varphi : \mathscr{U} \subset M \longrightarrow \varphi(\mathscr{U}) \subset \mathbb{C}^n ,$$

between an open neighborhood $\mathscr{U}$ of $\overline{D}$ and an open subset of $\mathbb{C}^n$, with the property that $D' = \varphi(D)$ is a strictly linearly convex domain $D' \subset \mathbb{C}^n$ and $\varphi_*(J_o)$ is sufficiently close to J_{st} in $\mathscr{C}^1$-norm, then D is a domain of circular type w.r.t. J_o with center $x \in D$ (for any $x \in D$!).

This result is essentially a generalization of the result for the case $D' = \mathbb{B}^n$, proved by Coupet, Gaussier and Sukhov in [15]. Its proof is also very close to a similar result, proved independently by Gaussier and Joo in [21].

The proof is technical and we just outline the key ingredients. The first thing to be done is to show that the boundary ∂D of a strictly (linearly) convex domain $D \subset (\mathbb{C}^n, J_{\mathrm{st}})$ is good for any pair (J_{st}, f) formed by the standard complex structure J_{st} and a stationary disk f of D through any $x_o \in D$ (here, "stationary" is in the usual sense, in the context of integrable complex structures). In this case, existence and uniqueness results are determined by Lempert's theory.

Secondly, for any $\mathrm{v}_o \in T_{x_o}D$, one considers the Riemann-Hilbert operator

$$\mathscr{R} = \mathscr{R}_{(J_{\mathrm{st}}, x_o, \mathrm{v}_o)} = (\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4, \mathscr{R}_5)$$

defined in (5.4) and the corresponding linearization

$$\mathfrak{R} = \mathfrak{R}_{(J_o, x_o, \mathrm{v}_o; \widetilde{f}_o, \lambda_o, \mu_o)} = (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5)$$

at a given triple $(\widetilde{f}_o, \lambda_o, \mu_o)$, corresponding to some lift $\widetilde{f}_o$ of a stationary disk f_o. One of the key points of the whole proof consists in showing that the operator $\widetilde{\mathfrak{R}} = (\mathfrak{R}_1, \mathfrak{R}_2)$, given by just the first two components of $\mathfrak{R}$, is surjective and with finite dimensional kernel. After this, if one consider the other three components $(\mathfrak{R}_3, \mathfrak{R}_4, \mathfrak{R}_5)$ (which correspond to the operators which fix the initial data and impose an additional normalizing condition), the resulting operator $\mathfrak{R}$ is "nailed down" to become a linear isomorphism.

The surjectivity of $\widetilde{\mathfrak{R}} = (\mathfrak{R}_1, \mathfrak{R}_2)$ is in fact a direct consequence of a result by Globevnik ([23]) and of general facts of the theory of Riemann-Hilbert problems. Such results can be used only if one is able to compute explicitly (and hence check whether they satisfy or not certain conditions) the so-called *partial indices* and the *Maslov index* of the conormal bundle along the boundaries of stationary disks of a strictly (linearly) convex domain $D \subset \mathbb{C}^n$. The computation of these indices is radically simplified by considering the so-called *flattening coordinates* of Lempert and Pang, in which a given stationary disk and the boundary nearby assume very simple expressions (see [38], Prop. 2.36 and Thm. 2.45).

We conclude this section, mentioning that there exists also the following "boundary version" for Theorem 5.5. As before, let $D \subset M$ be a smoothly bounded, relatively compact, strongly pseudoconvex domain in an almost complex manifold (M, J_o). For a fixed point $x_o \in \partial D$, consider a Riemannian metric $< \cdot, \cdot >$ around

x_o, which is Hermitian w.r.t. J_o, a normal vector ν to ∂M at x_o, pointing inwards, and for any $a > 0$, let us denote by $\mathscr{C}^{(a)}$ the cone

$$\mathscr{C}^{(a)} = \{ \, \mathrm{v} \in T_{x_o}M \, : < \mathrm{v}, \nu >> a \, \} \; .$$

With the same techniques of Theorem 5.5, one can prove the following:

Theorem 5.6 ([43]) *Assume that there exists a diffeomorphism*

$$\varphi : \mathscr{U} \subset M \longrightarrow \varphi(\mathscr{U}) \subset \mathbb{C}^n \; ,$$

between an open neighborhood $\mathscr{U}$ of $\overline{D}$ and an open subset of $\mathbb{C}^n$, with the property that $D' = \varphi(D)$ is a strictly linearly convex domain $D' \subset \mathbb{C}^n$ and that $\varphi_*(J_o)$ is sufficiently close to J_{st} in $\mathscr{C}^1$-norm. Then for any $x_o \in \partial D$ and $a > 0$, there exists a foliation by stationary disks of a subdomain $D^{(a)}_{(x_o)} \subset D$, in which all disks map $1 \in \partial\Delta$ into $x_o \in \partial D$ and have boundary tangent vector at x_o contained in the cone $\mathscr{C}^{(a)}$.

For the definition of *boundary tangent vector* and a more detailed description of the subdomain $D^{(a)}_{(x_o)} \subset D$, see [43].

5.2 *Almost complex domains of circular type and normal forms*

The project of this section can be roughly described as follows:

- Try to reproduce the steps, performed in the study of stationary disks and Monge-Ampère foliations of complex domains, in the new wider context of domains in *almost* complex manifolds.
- Show that such steps can in fact be performed for a very large class of almost complex domains and give useful information about pluripotential theory on almost complex domains.

As usual, let $D \subset (M,J)$ be a smooth, relatively compact domain in an almost complex manifold and, for any point x_o of an almost complex domain $D \subset (M,J)$, denote by $\mathscr{F}^{(x_o)}$ the family of stationary disks of D with $f(0) = x_o$. Imitating the definition of circular domains in complex manifolds, we introduce the following notion.

Definition 5.7 ([43]) We say that $\mathscr{F}^{(x_o)}$ is a *foliation of circular type* if:

i) for any $\mathrm{v} \in T_{x_o}D$, there exists a unique disk $f^{(\mathrm{v})} \in \mathscr{F}^{(x_o)}$ with $f^{(\mathrm{v})}_*\left(\left.\frac{\partial}{\partial x}\right|_0 \right) = \mu \cdot \mathrm{v}$ for some $0 \neq \mu \in \mathbb{R}$;
ii) for a fixed identification $(T_{x_o}D, J_{x_o}) \simeq (\mathbb{C}^n, J_{\mathrm{st}})$, the map from the blow up $\widetilde{\mathbb{B}}^n$ at 0 of $\mathbb{B}^n$ to the blow up $\widetilde{D}$ at x_o of D

$$\widetilde{E} : \widetilde{\mathbb{B}}^n \subset \widetilde{\mathbb{C}}^n \longrightarrow \widetilde{D} \;, \qquad \widetilde{E}(\mathrm{v},[\mathrm{v}]) \stackrel{\mathrm{def}}{=} \widetilde{f^{(\mathrm{v})}}(|\,\mathrm{v}\,|) \tag{5.9}$$

is smooth, extends smoothly up to the boundary and determines a diffeomorphism between the boundaries $\widetilde{E}|_{\partial\widetilde{\mathbb{B}}^n} : \partial\widetilde{\mathbb{B}}^n = \partial\mathbb{B}^n \longrightarrow \partial\widetilde{D} = \partial D$.

The point x_o is called *center of the foliation* and $\widetilde{E} : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{D}$ is called *(generalized) Riemann map of* (D, x_o). Any domain $D \subset (M, J)$ admitting a foliation of circular type centered at x_o is called *almost complex domain of circular type with center* x_o.

As in the integrable case, consider the blow-down map

$$\pi : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{B}^n$$

and the (uniquely defined) map $E : \mathbb{B}^n \longrightarrow D$ such that the following diagram commutes

$$\begin{tikzcd}
\widetilde{\mathbb{B}}^n \arrow[r, "\widetilde{E}"] \arrow[d, "\pi"'] & \widetilde{D} \arrow[d, "\pi"] \\
\mathbb{B}^n \arrow[r, "E"] & D
\end{tikzcd}$$

The map E defined in this way is $\mathscr{C}^\infty$ on $\mathbb{B}^n \setminus \{0\}$ and it is continuous at 0. However, in general, E is not smooth at 0. Nevertheless, one can consider a new differentiable structure on $\mathbb{B}^n$, formed by the atlas of coordinate charts of the form

$$\xi' : \mathscr{U} \longrightarrow \mathbb{R}^{2n} , \qquad \xi' = \xi \circ E ,$$

where the $\xi = (x^i)$ are charts of the differentiable structure of $D \subset M$. By construction, these charts overlap smoothly with the standard coordinates of $\mathbb{R}^{2n}$ on any open subsets of $\mathbb{B}^n \setminus \{0\}$. Hence their restrictions on $\mathbb{B}^n \setminus \{0\}$ belong to the standard differentiable structure of $\mathbb{B}^n$. On the other hand, when E is not smooth at 0, they cannot smoothly overlap with standard coordinates in neighborhoods of 0. This means that *they give a non-standard differentiable structure on* $\mathbb{B}^n$*, which coincides with the standard one only on* $\mathbb{B}^n \setminus \{0\}$.

In the following, we always implicitly consider on $\mathbb{B}^n$ such new differentiable structure. Notice that, by construction, the map $E : \mathbb{B}^n \longrightarrow D$ is smooth also at 0 if $\mathbb{B}^n$ is endowed with such non-standard differentiable structure.

Now, let us consider on $\mathbb{B}^n$ the almost complex structure J' defined by

$$J' = E^*(J) .$$

We stress once again that *the tensor field* J' *is smooth over the whole* $\mathbb{B}^n$*, provided that one considers the differentiable structure on* $\mathbb{B}^n$ *defined above; w.r.t. the stan-*

dard differentiable structure, J' is smooth only on $\mathbb{B}^n \setminus \{0\}$ and it is possibly non well-defined at $0 \in \mathbb{B}^n$.

The pair $(\mathbb{B}^n, J')$ is called *normal form of the almost complex domain of circular type* (D, J). After all necessary verifications, one can conclude that the following perfect analogue of the situation in the integrable case, holds true: *Any almost complex domain $D \subset (M, J)$ of circular type is (J, J')-biholomorphic to its normal form $(\mathbb{B}^n, J')$ through a map that sends the J-stationary disks of D into the straight radial disks of $\mathbb{B}^n$, which are therefore J'-stationary disks.*

This shows that the analysis of almost complex domains can be reduced to the study of almost complex structures on $\mathbb{B}^n$ with the above properties, i.e., such that the straight radial disks are J'-stationary.

Consider the distribution $\mathscr{Z} \subset T\mathbb{B}^n$ defined in (3.1). We have the following.

Theorem 5.8 *A pair $(\mathbb{B}^n, J)$, formed by $\mathbb{B}^n$ endowed with a non-standard differentiable structure, coinciding with the standard one on $\mathbb{B}^n \setminus \{0\}$, and an almost complex structure J which is smooth w.r.t. such differentiable structure, is a domain of circular type in normal form if and only if*

i) $\mathscr{Z}$ is J-invariant (i.e., $J\mathscr{Z} = \mathscr{Z}$);
ii) the straight radial disks of $\mathbb{B}^n$ are J-stationary;
iii) the non-standard differentiable structure is such that the blow-up $\widetilde{\mathbb{B}}^n$ of $\mathbb{B}^n$ at 0, determined by J, is equivalent, as differentiable manifold, to the blow-up determined by the standard complex structure.

In the following, the almost complex structures J on $\mathbb{B}^n$, satisfying (i) - (iii) of previous theorem, will be called *L-almost complex structure*.

It is important to observe that (ii) holds if and only if, in suitable systems of coordinates, the components of J belong to the range of a Fredholm operator, i.e., they belong to a space, which is finite codimensional in an appropriate Hilbert space, and can be characterized by a finite number of equations. This fact has a pair of interesting consequences, namely that:

a) the class of L-almost complex structure is in practice a very large class;
b) such class naturally includes two smaller classes, characterized by very simple conditions, which are very useful to construct a number of interesting examples.

The definitions and the analysis of such smaller classes are the contents of next section.

5.3 "Nice" and "very nice" L-almost complex structures

Consider the distributions $\mathscr{Z}$ and $\mathscr{H}$ on $\mathbb{B}^n$ defined in (3.1) and (3.2). We recall that:

– denoting, as usual, $\tau_o(z) = \|z\|^2$ and $d^c_{\mathrm{st}} = J^*_{\mathrm{st}} \circ d \circ J^*_{\mathrm{st}}$, one has that

$$\begin{cases} J_{\mathrm{st}} \mathscr{Z}_z = \mathscr{Z}_z \\ \\ \mathscr{Z}_z = \ker dd^c_{\mathrm{st}} \log \tau_o \end{cases} \qquad \text{for any } z \in \mathbb{B}^n \setminus \{0\} \ ;$$

– $\mathscr{Z}$ and $\mathscr{H}$ are not only orthogonal w.r.t. the Euclidean metric but also w.r.t. the J_{st}-invariant 2-form $dd^c_{J_{\mathrm{st}}} \tau_o$;
– for any $z \in \mathbb{B}^n \setminus \{0\}$, the subspace $\mathscr{H}_z \subset T_z\mathbb{B}^n$ coincides with the J_{st}-holomorphic tangent space of the sphere $S_c = \{\ \tau_o = c\ \}$, $c = \tau_o(z)$:
– for any tangent space $T_z\mathbb{B}^n$, $z \neq 0$, the complexification $T^{\mathbb{C}}_z\mathbb{B}^n$ decomposes into the direct sum

$$T^{\mathbb{C}}_z\mathbb{B}^n = \mathscr{Z}^{\mathbb{C}}_z \oplus \mathscr{H}^{\mathbb{C}}_z = \left(\mathscr{Z}^{10}_z \oplus \mathscr{Z}^{01}_z\right) \oplus \left(\mathscr{H}^{10}_z \oplus \mathscr{H}^{01}_z\right) \ ,$$

where we denoted by $\mathscr{Z}^{01}_z$, $\mathscr{H}^{01}_z$ the $(-i)$-eigenspaces of J_{st} in $\mathscr{Z}^{\mathbb{C}}_z$ and $\mathscr{H}^{\mathbb{C}}_z$, and by $\mathscr{Z}^{10}_z$, $\mathscr{H}^{10}_z$ their complex conjugates (which are the $(+i)$-eigenspaces).

Now, an arbitrary almost complex structure J on $\mathbb{B}^n$ is uniquely determined by the corresponding distribution of $(-i)$-eigenspaces $(T_z\mathbb{B}^n)^{01}_J$ in $T^{\mathbb{C}}_x\mathbb{B}^n$. Generically, these eigenspaces are determined by a tensor field $\varphi \in \mathrm{Hom}(T^{01}\mathbb{B}^n, T^{10}\mathbb{B}^n)$ such that

$$(T_z\mathbb{B}^n)^{01}_J = T^{01}_z\mathbb{B}^n + \varphi(T^{01}_z\mathbb{B}^n) \ ,$$

where, as before, we denoted $T^{01}_z\mathbb{B}^n = (T_z\mathbb{B}^n)^{01}_{J_{\mathrm{st}}}$ and $T^{10}_z\mathbb{B}^n = (T_z\mathbb{B}^n)^{10}_{J_{\mathrm{st}}}$. If we consider the decomposition of ϕ as a sum of the form

$$\phi = \phi^{\mathscr{Z}} \oplus \phi^{\mathscr{H}} \oplus \phi^{\mathscr{Z},\mathscr{H}} \oplus \phi^{\mathscr{H},\mathscr{Z}}$$

with

$$\phi^{\mathscr{Z}} \in \mathrm{Hom}(\mathscr{Z}^{01}, \mathscr{Z}^{10}) \ , \quad \phi^{\mathscr{H}} \in \mathrm{Hom}(\mathscr{H}^{01}, \mathscr{H}^{10}) \ , \quad \phi^{\mathscr{Z},\mathscr{H}} \in \mathrm{Hom}(\mathscr{Z}^{01}, \mathscr{H}^{10}) \ ,$$
$$\phi^{\mathscr{H},\mathscr{Z}} \in \mathrm{Hom}(\mathscr{H}^{01}, \mathscr{Z}^{10}) \ ,$$

we have that the $(-i)$-eigenspaces $(T_z\mathbb{B}^n)^{01}_J$ can be written as

$$\begin{aligned}(T_z\mathbb{B}^n)^{01}_J = &\Big(\ \mathscr{Z}^{01} + \phi^{\mathscr{Z}}_z(\mathscr{Z}^{01}) + \phi^{\mathscr{Z},\mathscr{H}}_z(\mathscr{Z}^{01})\Big) + \\ &+ \Big(\ \mathscr{H}^{01} + \phi^{\mathscr{H}}_z(\mathscr{H}^{01}) + \phi^{\mathscr{H},\mathscr{Z}}_z(\mathscr{H}^{01})\Big).\end{aligned} \tag{5.10}$$

We observe that J satisfies condition (i) of Theorem 5.8 (i.e., $J(\mathscr{Z}) \subset \mathscr{Z}$ and $J|_{\mathscr{Z}} = J_{\mathrm{st}}|_{\mathscr{Z}}$) if and only if the components $\varphi^{\mathscr{Z}}$ and $\varphi^{\mathscr{Z},\mathscr{H}}$ are identically equal to 0, that is

$$\varphi = \varphi^{\mathscr{H}} + \varphi^{\mathscr{H},\mathscr{Z}} \ .$$

In such class of almost complex structures J on $\mathbb{B}^n$, it is very convenient to consider the following conditions.

1) J is called *nice* if the corresponding deformation tensor φ is, in addition, of the form
$$\varphi = \varphi^{\mathscr{H}} .$$
This is equivalent to assume that the distribution $\mathscr{H}$ is J-invariant.
2) J is called *very nice* if
$$\varphi = \varphi^{\mathscr{H}} \qquad \text{and} \qquad \mathscr{L}_{Z^{0,1}} \varphi^{\mathscr{H}} = 0 .$$
This assumption corresponds to require that J is nice and that the deformation tensor φ depends J_{st}-holomorphically on the complex parameter that describe the straight radial disks of $\mathbb{B}^n$.

A geometric motivation for considering the notion of "very nice structures" comes from the following. It is well known that, in case of *integrable* complex structures, there exists a strict relation between stationary disks and Kobayashi extremal disks. This is a fact that goes back to the ideas of Poletski ([49]) and Lempert ([31]), which showed that, under appropriate regularity conditions, the stationary disks are the solutions to the Euler-Lagrange conditions for the extremal Kobayashi disks (they are the critical point of a suitable functional!). In fact, by the results of [31] we know that the two notions agree for the strictly linearly convex domains in $\mathbb{C}^n$.

Now, it is important to have in mind that, for generic almost complex domains, *these two notions – stationarity and extremality – are no longer related*. Counterexamples have been recently exhibited by Gaussier and Joo in [21]. The authors determined also some conditions, which are sufficient for a stationary disk to be also extremal and which can be described as follows.

Let us first recall a few concepts related with the geometry of the tangent bundle of a manifold M. We recall that the *vertical distribution in $T(TM)$* is the subbundle of $T(TM)$ defined by
$$T^V(TM) = \bigcup_{(x,\mathrm{v})\in TM} T^V_{(x,\mathrm{v})}M \ , \qquad T^V_{(x,\mathrm{v})}M = \ker \pi_*|_{(x,\mathrm{v})} .$$
For any $x \in M$, let us denote by $(\cdot)^V : T_xM \longrightarrow T^V_{(x,\mathrm{v})}M$ the map
$$\left(\mathrm{w}^i \frac{\partial}{\partial x^i}\bigg|_x \right)^V \overset{\mathrm{def}}{=} \mathrm{w}^i \frac{\partial}{\partial q^i}\bigg|_{(x,\mathrm{v})} .$$
It is possible to check that this map does not depend on the choice of coordinates and that it determines a natural map from TM to $T(TM)$ (see [59]). For any $\mathrm{w} \in TM$, the corresponding vector $\mathrm{w}^V \in T(TM)$ is called *vertical lift of* w.

Definition 5.9 ([22, 44]) Let $f : \overline{\Delta} \longrightarrow M$ be a $\mathscr{C}^{\alpha,\varepsilon}$, J-holomorphic embedding with $f(\partial\Delta) \subset \partial D$. We call *infinitesimal variation of* f any $\mathbb{J}$-holomorphic map $W : \overline{\Delta} \longrightarrow TM$ of class $\mathscr{C}^{\alpha-1,\varepsilon}$ with $\pi \circ W = f$ (here, $\pi : TM \longrightarrow M$ is the natural projection). An infinitesimal variation W is called *attached to* ∂D *and with fixed center* if

a) $\alpha(W_\zeta) = 0$ for any $\alpha \in \mathscr{N}_{f(\zeta)}$, $\zeta \in \partial\Delta$,
b) $W|_0 = 0$.

It is called *with fixed central direction* if in addition it satisfies

c) $W_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) \in T^V_{W_0}(TM)$ and it is equal to $\lambda\left(f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right)\right)^V$ for some $\lambda \in \mathbb{R}$.

The disk f is called *Kobayashi critical* if for any infinitesimal variation W, attached to ∂D and with fixed central direction, one has $W_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Big|_0\right) = 0$.

Such definition is motivated by the fact that, when $f^{(t)} : \overline{\Delta} \longrightarrow M$, $t \in]-a,a[$, is a smooth 1-parameter family of J-holomorphic disks of class $\mathscr{C}^{\alpha,\varepsilon}$ with $f^{(0)} = f$, then $W = \frac{df^{(t)}}{dt}\Big|_{t=0}$ is a variational field on f. Moreover, if $f^{(t)}$ is such that, for all $t \in]a,a[$

$$f^{(t)}(\partial\Delta) \subset \partial D\ ,\ \ f^{(t)}(0) = f(0)\ ,\ \ f^{(t)}_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Bigg|_0\right) \in \mathbb{R} f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Bigg|_0\right)\ , \tag{5.11}$$

then W satisfies (a) - (c). On the other hand, a disk f is a *locally extremal disk* if for any J-holomorphic disk $g : \overline{\Delta} \longrightarrow M$ of class $\mathscr{C}^{\alpha,\varepsilon}$, with image contained in some neighborhood of $f(\overline{\Delta})$ and such that, for some $\lambda \in \mathbb{R}$,

$$g(\partial\Delta) \subset \partial D\ , \quad g(0) = f(0) = x_o\ , \quad g_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Bigg|_0\right) = \lambda f_*\left(\frac{\partial}{\partial \operatorname{Re}\zeta}\Bigg|_0\right)\ ,$$

then $\lambda \leq 1$. It is known that the notions of "Kobayashi critical" and "locally extremal Kobayashi disk" are tightly related. In fact, *any locally extremal disk f, with $f(\partial\Delta) \subset \partial D$, is Kobayashi critical. Conversely, when $D \subset \mathbb{C}^n$ is strictly convex around $f(\overline{\Delta})$ and J is sufficiently close to J_{st}, any Kobayashi critical disk f is locally extremal* ([21, 22, 44]).

Next theorem gives conditions that imply the equality between stationary and critical disks and will be used in the sequel. It is a refinement of a result and arguments given in [20] (see [44]). In this statement, $f : \overline{\Delta} \longrightarrow M$ is a J-holomorphic embedding, of class $\mathscr{C}^{\alpha,\varepsilon}$ with $f(\partial\Delta) \subset \partial D$, and $\mathfrak{Var}_o(f)$ denotes the class of infinitesimal variations of f attached to ∂D and with fixed center.

Theorem 5.10 *Assume that $D \subset M$ is of the form $D = \{\ \rho < 0\ \}$ for some J-plurisubharmonic ρ and that $\mathfrak{Var}_o(f)$ contains a $(2n-2)$-dimensional J-invariant vector space, generated by infinitesimal variations e_i, Je_i, $1 \leq i \leq n-1$, such that the maps*

$$\zeta^{-1} \cdot e_i(\zeta)\ ,\ \zeta^{-1} \cdot Je_i(\zeta) : \overline{\Delta} \longrightarrow TM$$

are of class $\mathscr{C}^{\alpha,\varepsilon}$ *on* $\overline{\Delta}$*. Assume also that, for any* $\zeta \in \overline{\Delta}$*, the set* $\{e_i(\zeta), Je_i(\zeta)\} \subset T_{f(\zeta)}M$ *span a subspace, which is complementary to* $T_{f(\zeta)}f(\Delta) \subset T_{f(\zeta)}M$*.*

Then f *is critical if and only if it is stationary.*

This result has a direct application in our case. Assume that $(\mathbb{B}^n, J)$ is an almost complex domain of circular type in normal form (i.e., such that J is an L-almost complex structure).

One can construct variations of the straight radial disks of $\mathbb{B}^n$, deforming them through the directions of $\mathscr{H}$ and obtain a special subspace $\widetilde{\mathfrak{V}} \subset \mathfrak{Var}_o(f)$ of infinitesimal variations for any given straight radial disk f. It turns out that when J is nice, (i.e., $\mathscr{H}$ is J-invariant),

$$J\widetilde{\mathfrak{V}} \subset \mathfrak{Var}_o(f) \qquad \Longleftrightarrow \qquad \mathscr{L}_{Z^{01}}J = 0\ ,$$

i.e., if and only if J is very nice. Combining this fact with the previous theorem, one is able to prove the following proposition which motivates the interest for “very nice structures”.

Proposition 5.11 *Let* $(\mathbb{B}^n, J)$ *be an almost complex domain of circular type in normal form. If* J *is very nice, the straight radial disks of* $\mathbb{B}^n$ *are not only stationary disks but also Kobayashi critical.*

6 Plurisubharmonic functions and pseudoconvex almost complex manifolds

Let (M,J) be an almost complex manifold and $\Omega^k(M)$, $k \geq 0$, the space of k-forms of M. We denote by $d^c : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ the classical d^c-operator

$$d^c\alpha = (-1)^k (J^* \circ d \circ J^*)(\alpha)\ ,$$

where J^* denotes the usual action of J on k-forms, i.e.,

$$J^*\beta(\mathrm{v}_1, \ldots, \mathrm{v}_k) \stackrel{\text{def}}{=} (-1)^k \beta(J\mathrm{v}_1, \ldots, J\mathrm{v}_k)$$

Let us recall, once again, that when J is integrable,

$$d^c = i(\overline{\partial} - \partial)\ , \quad \partial\overline{\partial} = \frac{1}{2i} dd^c\ , \quad dd^c = -d^c d$$

and that $dd^c u$ is a J-Hermitian 2-form for any $\mathscr{C}^2$-function u. We stress the fact that, *when* J *is not integrable,* $d^c d \neq -dd^c$ *and the 2-forms* $dd^c u$*, determined by the functions* $u \in \mathscr{C}^2(M)$*, are usually not* J*-Hermitian*. In fact, one has that

$$dd^c u(JX_1, X_2) + dd^c u(X_1, JX_2) = 4N_{X_1X_2}(u) \ , \tag{6.1}$$

where $N_{X_1X_2}$ is the Nijenhuis tensor evaluated on X_1, X_2 and is – of course – in general non zero. This fact suggests the following definition.

Definition 6.1 Let $u : \mathscr{U} \subset M \longrightarrow \mathbb{R}$ be of class $\mathscr{C}^2$. We call *J-Hessian of u at x* the symmetric form $Hess(u)_x \in S^2 T_x M$, whose associated quadratic form is $\mathscr{L}(u)_x(\mathrm{v}) = dd^c u(\mathrm{v}, J\mathrm{v})_x$. By polarization formula and (6.1), one has that, for any v, w $\in T_x M$,

$$\begin{aligned} Hess(u)_x(\mathrm{v},\mathrm{w}) &= \frac{1}{2}\left(dd^c u(\mathrm{v}, J\mathrm{w}) + dd^c u(\mathrm{w}, J\mathrm{v})\right)\Big|_x = \\ &= dd^c u(\mathrm{v}, J\mathrm{w})_x - 2N_{\mathrm{v}\,\mathrm{w}}(u) \ . \end{aligned} \tag{6.2}$$

We remark that $Hess(u)_x$ is not only symmetric, but also J-Hermitian, i.e.,

$$Hess(u)_x(J\mathrm{v}, J\mathrm{w}) = Hess(u)_x(\mathrm{v},\mathrm{w}) \qquad \text{for any} \quad \mathrm{v},\mathrm{w}$$

and it is associated with the Hermitian antisymmetric tensor

$$Hess(u)(J\cdot,\cdot) = \frac{1}{2}\left(dd^c u(\cdot,\cdot) + dd^c u(J\cdot, J\cdot)\right) = \frac{1}{2}\left(dd^c u + J^* dd^c u\right). \tag{6.3}$$

The *Levi form of u at x* is the quadratic form

$$\mathscr{L}(u)_x(\mathrm{v}) = dd^c u(\mathrm{v}, J\mathrm{v})|_x \ .$$

The operator dd^c defined above turns out to be suitable to study plurisubhamonicity on almost complex manifolds. It has been used for instance by Pali in [37] for his study of positivity questions and in a very recent work of Harvey and Lawson ([25]), using a completely different point of view involving viscosity approach, to provide a satisfactory *weak* pluripotential theory in the almost complex setting. We will further give evidence that it is appropriate to define the almost complex Monge-Ampère operator. Finally, we point out that Pliś ([48]) uses it to study the inhomogeneous almost complex Monge-Ampère equation.

An upper semicontinuous function $u : \mathscr{U} \subset M \longrightarrow \mathbb{R}$ is called *J-plurisubharmonic* if

$$u \circ f : \Delta \longrightarrow \mathbb{R}$$

is subharmonic for any J-holomorphic disk $f : \Delta \longrightarrow \mathscr{U} \subset M$. As for complex manifolds, for any $u \in \mathscr{C}^2(\mathscr{U})$ one has that

u is J-plurisubharmonic

if and only if

$$\mathscr{L}(u)_x(\mathrm{v}) = Hess(u)_x(\mathrm{v},\mathrm{v}) \geq 0 \qquad \textit{for any} \ \ x \in \mathscr{U} \ \ \textit{and} \ \ \mathrm{v} \in T_x M \ .$$

This motivates the following generalizations of classical notions. In the following, for any $\mathscr{U} \subset M$, the symbol $\mathrm{Psh}(\mathscr{U})$ denotes the class of J-plurisubharmonic functions on $\mathscr{U}$.

Definition 6.2 Let (M,J) be an almost complex manifold and $\mathscr{U} \subset M$ an open subset. We say that $u \in \mathrm{Psh}(\mathscr{U})$ is *strictly J-plurisubharmonic* if:

a) $u \in L^1_{\mathrm{loc}}(\mathscr{U})$;
b) for any $x_o \in \mathscr{U}$ there exists a neighborhood $\mathscr{V}$ of x_o and $v \in \mathscr{C}^2(\mathscr{V}) \cap \mathrm{Psh}(\mathscr{V})$ for which $Hess(v)_x$ is positive definite at all points and $u - v$ is in $\mathrm{Psh}(\mathscr{V})$.

In particular, $u \in \mathrm{Psh}(\mathscr{U}) \cap \mathscr{C}^2(\mathscr{U})$ is strictly plurisubharmonic if and only if $Hess(u)_x$ is positive definite at any $x \in \mathscr{U}$.

The almost complex manifold (M,J) is called *strongly pseudoconvex* (or *Stein*) *manifold* if it admits a $\mathscr{C}^2$ -strictly plurisubharmonic exhaustion $\tau : M \longrightarrow]-\infty,\infty[$.

6.1 Maximal plurisubharmonic functions

The J-plurisubharmonic functions share most of the basic properties of classical plurisubharmonic functions. In particular, as it occurs for the domains in complex manifolds, for any open domain $\mathscr{U} \subset (M,J)$, the class $\mathrm{Psh}(\mathscr{U})$ is a convex cone and for any given $u_i \in \mathrm{Psh}(\mathscr{U})$ and $\lambda_i \in \mathbb{R}$, also

$$u = \sum_{i=1}^{n} \lambda_i u_i \qquad \text{and} \qquad u' = \max\{\, u_1, \ldots, u_n \,\}$$

are in $\mathrm{Psh}(\mathscr{U})$. It is therefore natural to consider the following notion of "maximal" J-plurisubharmonic functions.

Definition 6.3 Let D be a domain in a strongly pseudoconvex almost complex manifold (M,J). A function $u \in \mathrm{Psh}(D)$ is called *maximal* if for any open $\mathscr{U} \subset\subset D$ and $h \in \mathrm{Psh}(\mathscr{U})$ satisfying the condition

$$\limsup_{z \to x} h(z) \leq u(x) \qquad \text{for all } x \in \partial\mathscr{U} \;, \tag{6.4}$$

one has that $h \leq u|_{\mathscr{U}}$.

The following characterization of maximal plurisubharmonic functions "nails down" the right candidate for what should be considered as "almost complex Monge-Ampère operator".

Theorem 6.4 *Let $D \subset M$ be a domain of a strongly pseudoconvex almost complex manifold (M,J) of dimension $2n$. A function $u \in \mathrm{Psh}(D) \cap \mathscr{C}^2(D)$ is maximal if and only if it satisfies*

$$(dd^c u + J^*(dd^c u))^n = 0\ . \tag{6.5}$$

Proof. Let $\tau : M \longrightarrow]-\infty, +\infty[$ be a $\mathscr{C}^2$ strictly plurisubharmonic exhaustion for M and assume that u satisfies (6.5). We need to show that for any $h \in \mathrm{Psh}(\mathscr{U})$ on an $\mathscr{U} \subset\subset D$ that satisfies (6.4), one has that $h \leq u|_{\mathscr{U}}$. Suppose not and pick $\mathscr{U} \subset\subset D$ and $h \in \mathrm{Psh}(\mathscr{U})$, so that (6.4) is true but there exists $x_o \in \mathscr{U}$ with $u(x_o) < h(x_o)$. Let $\lambda > 0$ so small that

$$h(x_o) + \lambda\,(\tau(x_o) - M) > u(x_o)\ , \qquad \text{where } M = \max_{y \in \overline{\mathscr{U}}} \tau(y)\ ,$$

and denote by $\widehat{h}$ the function

$$\widehat{h} \stackrel{\text{def}}{=} h + \lambda(\tau - M)|_{\mathscr{U}}\ . \tag{6.6}$$

By construction, $\widehat{h} \in \mathrm{Psh}(\mathscr{U})$, satisfies (6.4) and $(\widehat{h} - u)(x_o) > 0$. In particular, $\widehat{h} - u$ achieves its maximum at some inner point $y_o \in \mathscr{U}$. Now, we remark that (6.5) is equivalent to say that, for any $x \in D$, there exists $0 \neq \mathrm{v} \in T_x M$ so that

$$(dd^c u + J^*(dd^c u))_x(\mathrm{v}, J\mathrm{v}) = Hess_x(u)(\mathrm{v}, \mathrm{v}) = 0\ . \tag{6.7}$$

Let $0 \neq \mathrm{v}_o \in T_{y_o} M$ be a vector for which (6.7) is true and let $f : \Delta \longrightarrow M$ be a J-holomorphic disk so that $f(0) = y_o$ and with

$$f_*\left(\left.\frac{\partial}{\partial x}\right|_0\right) = \mathrm{v}_o\ , \qquad f_*\left(\left.\frac{\partial}{\partial y}\right|_0\right) = f_*\left(J_{\mathrm{st}}\left.\frac{\partial}{\partial x}\right|_0\right) = J\,\mathrm{v}_o\ .$$

Then, consider the function $G : \Delta \longrightarrow \mathbb{R}$ defined by

$$G \stackrel{\text{def}}{=} \widehat{h} \circ f - u \circ f = h \circ f + (\lambda\tau - \lambda M - u) \circ f\ . \tag{6.8}$$

We claim that there exists a disk $\Delta_r = \{|\zeta| < r\}$ such that $G|_{\Delta_r}$ is subharmonic. In fact, since τ is $\mathscr{C}^2$ and strictly plurisubharmonic and $Hess(u)_{y_o}(\mathrm{v}_o, \mathrm{v}_o) = 0$, we have that

$$0 < Hess((\lambda\tau - \lambda M - u))_{y_o}(\mathrm{v}_o, \mathrm{v}_o) = 2i\,\partial\overline{\partial}((\lambda\tau - \lambda M - u) \circ f)\Big|_0\ .$$

Hence, by continuity, there exists $r > 0$ so that

$$0 < 2i\,\partial\overline{\partial}((\lambda\tau - \lambda M - u) \circ f)\Big|_\zeta \qquad \text{for any} \quad \zeta \in \overline{\Delta_r}\ .$$

It follows that $(\lambda\tau - \lambda M - u) \circ f|_{\Delta_r}$ is strictly subharmonic and that $G|_{\Delta_r}$ is subharmonic, being sum of subharmonic functions. At this point, it suffices to observe that, since y_o is a point of maximum for $\widehat{h} - u$ on $f(\Delta) \subset \mathscr{U}$, then $0 = f^{-1}(y_o) \in \Delta_r$ is an inner point of maximum for $G|_{\Delta_r}$. In fact, from this and the maximum principle, we get that $G|_{\Delta_r}$ is constant and hence that $h\circ f|_{\Delta_r}$ is $\mathscr{C}^2$ with $2i\,\partial\overline{\partial}(h\circ f)\Big|_{\Delta_r} < 0$, contradicting the hypothesis on subharmonicity of $h\circ f$.

Conversely, assume that $u \in \mathscr{C}^2(D)\cap \mathrm{Psh}(D)$ is maximal, but that there exists $y_o \in D$ for which $Hess_{y_o}(u)(\mathrm{v},\mathrm{v}) > 0$ for any $0 \neq \mathrm{v} \in T_{y_o}M$ and consider the following well known result (see e.g. [18]).

Lemma 6.5 *For any $\varepsilon > 0$, there exists a relatively compact neighborhood $\mathscr{U}$ of y_o, such that $(\mathscr{U}, J)$ is (J,J')-biholomorphic to $(\mathbb{B}^n, J')$ for some J' such that $\|J' - J_{\mathrm{st}}\|_{\mathbb{B}^n,\,\mathscr{C}^2} < \varepsilon$.*

Due to this, we may assume that $\tau = \tau_o\circ\varphi$, with $\tau_o(z) = \|z\|^2$, is a $\mathscr{C}^2$ strictly J-plurisubharmonic exhaustion on $\mathscr{U}$, tending to 1 at the points of $\partial\mathscr{U}$. Hence, there is a constant $c > 0$ such that

$$Hess_x(u + c(1-\tau))(\mathrm{v},\mathrm{v}) = Hess_x(u)(\mathrm{v},\mathrm{v}) - cHess_x(\tau)(\mathrm{v},\mathrm{v}) \geq 0\ ,$$

for all $x \in \mathscr{U}$ and $\mathrm{v} \in T_xM \simeq \mathbb{R}^{2n}$ with $\|\,\mathrm{v}\,\| = 1$. This means that

$$\widehat{h} \overset{\mathrm{def}}{=} (u + c(1-\tau))|_{B_{y_o}(r)}$$

is in $\mathscr{C}^2(\mathscr{U})\cap \mathrm{Psh}(\mathscr{U})$, satisfies (6.4) and, by maximality of u, satisfies $\widehat{h} \leq u$ at all points of $\mathscr{U}$. But there is also an $\varepsilon > 0$ such that $\emptyset \neq \tau^{-1}([0, 1-\varepsilon[) \subsetneq \mathscr{U}$ and hence such that, on this subset, $\widehat{h} \geq u + c\varepsilon > u$, contradicting the maximality of u. $\square$

6.2 Green functions of nice circular domains

The results of previous section motivate the following generalized notion of Green functions.

Definition 6.6 Let D be a domain in a strongly pseudoconvex, almost complex manifold (M,J). We call *almost pluricomplex Green function with pole at $x_o \in D$* an exhaustion $u : \overline{D} \longrightarrow [-\infty, 0]$ such that

i) $u|_{\partial D} = 0$ and $u(x) \simeq \log\|x - x_o\|$ when $x \to x_o$, for some Euclidean metric $\|\cdot\|$ on a neighborhood of x_o;
ii) it is J-plurisubharmonic;
iii) it is a solution of the generalized Monge-Ampere equation $(dd^c u + J^*(dd^c u))^n = 0$ on $D\setminus\{x_o\}$.

Notice that, if a Green function with pole x_o exists, by a direct consequence of property of maximality (Theorem 6.4), it is unique.

Consider an almost complex domain D of circular type in (M,J) with center x_o and denote by $\widetilde{E} : \widetilde{\mathbb{B}}^n \longrightarrow \widetilde{D}$ the corresponding Riemann map. We call *standard* *exhaustion of D* the map

$$\tau_{(x_o)} : D \longrightarrow [0,1[\ , \qquad \tau(x) = \begin{cases} |\widetilde{E}^{-1}(x)|^2 & \text{if } x \neq 0\ , \\ \\ 0 & \text{if } x = x_o\ . \end{cases}$$

so that, when D is in normal form, i.e., when $D = (\mathbb{B}^n, J)$ with J almost L-complex structure, its standard exhaustion is just $\tau_o(z) = \|z\|^2$.

Proposition 6.7 *Let D be a domain of circular type in (M,J) with center x_o and standard exhaustion $\tau_{(x_o)}$. If $u = \log \tau_{(x_o)}$ is J-plurisubharmonic, then u is an almost pluricomplex Green function with pole at x_o.*

Proof. With no loss of generality, we may assume that the domain is in normal form, i.e., $D = (\mathbb{B}^n, J)$ and $\tau_{(x_o)}(z) = \tau_o(x) = \|x\|^2$. Since τ_o is smooth on $\mathbb{B}^n \setminus \{0\}$ and $u = \log \tau_o$ is J-plurisubharmonic, we have that $Hess(u)_x \geq 0$ for any $x \neq 0$. On the other hand, for any straight disk $f : \Delta \longrightarrow \mathbb{B}^n$ of the form $f(\zeta) = \mathrm{v} \cdot \zeta$, we have that $u \circ f$ is harmonic and $Hess(u)_{f(\zeta)}(\mathrm{v}, \mathrm{v}) = 0$ for any $\zeta \neq 0$. This means that $Hess(u)_x \geq 0$ has at least one vanishing eigenvalue at any point of $\mathbb{B}^n \setminus \{0\}$ and that (6.5) is satisfied. The other conditions of Definition 6.6 can be checked directly from definitions. □

When J is integrable, the standard exhaustion $u = \log \tau_{(x_o)}$ of the normal form of a domain of circular type is automatically plurisubharmonic ([42]), but *in the almost complex case, this is no longer true,* even for small deformations of the standard complex structure. Though this fact is known, it is important to try and understand why it happens. In the next section, we will illustrate how it is easy to produce illuminating examples using our deformation arguments and we provide hints on how to avoid such pathologies.

6.3 *A counterexample and pluricomplex Green functions of nice domains.*

On the blow up $\pi : \widetilde{B}^2 \longrightarrow B^2$ of the unit ball $B^2 \subset \mathbb{C}^2$ (defined in the usual way, via the standard complex structure J_{st}) consider the vector fields

$$Z\ , \quad J_{\mathrm{st}} Z\ , \quad E\ , \quad J_{\mathrm{st}} E\ ,$$

where Z is the lift on $\widetilde{\mathbb{B}}^n$ of the real vector field $\mathrm{Re}\left(z^i \frac{\partial}{\partial z^i} \right)$ on $\mathbb{B}^n$ and E is any vector field in the distribution $\mathscr{H}$ that satisfies the conditions

$$[Z,E] = [J_{\mathrm{st}} Z, E] = 0\ , \qquad [E, J_{\mathrm{st}} E] = -J_{\mathrm{st}} Z\ . \tag{6.9}$$

The standard holomorphic bundle $T^{10}\widetilde{B}^2$ is generated at all points by the complex vector field

$$Z^{10} = Z - iJ_{\mathrm{st}}Z \; ,$$

which determines the "radial" distribution, and by the complex vector field

$$E^{10} = E - iJ_{\mathrm{st}}E \; ,$$

which determines the holomorphic tangent bundles of the spheres $S_c = \{\ \tau_o(z) = \|z\| = c\ \}$.

Let us also denote by $(E^{10*}, E^{01*}, Z^{10*}, Z^{01*})$ the field of complex coframes, which is dual to the complex frame field $(E^{10}, E^{01} = \overline{E^{10}}, Z^{10}, Z^{01} = \overline{Z^{10}})$ at all points.

Consider now a smooth real valued function $h : \widetilde{\mathbb{B}}^n \longrightarrow \mathbb{R}$ such that

– on each sphere S_c, the restriction $h|_{S_c}$ is constant,
– $h \equiv 0$ on an open neighborhood of $\pi^{-1}(0) = \mathbb{C}P^1$,

and let $\varphi \in \mathrm{Hom}(\mathscr{H}^{01}, \mathscr{Z}^{10} + \mathscr{H}^{10})$ be the deformation tensor

$$\varphi_z = h(z) Z^{10}_z \otimes E^{01}{}^*_z \; .$$

The almost complex structure J, corresponding to φ, is uniquely determined by the J-holomorphic spaces

$$T^{10}_{J_z}\widetilde{\mathbb{B}}^n = \mathbb{C}Z^{10}_z \oplus \mathbb{C}\widetilde{E}^{10}_z \quad \text{where } \; \widetilde{E}^{10}_z \stackrel{\text{def}}{=} E^{10}_z + h(z) Z^{01}_z \; ,$$

By direct inspection, it is not hard to check that J is an almost L-complex structure and that $(\widetilde{\mathbb{B}}^n, J)$ is an almost complex domain of circular type in normal form ([44]). But we also have the following crucial fact.

Fact: *If $h \not\equiv 0$, the function $u = \log \tau_o$ is not J-plurisubharmonic.*

Indeed, using the definition of J and (6.9), one computes

$$Hess(\widetilde{E}^{10}, \widetilde{E}^{10}) = 2(1 + 2hh_Z) \; ,$$

$$Hess(\widetilde{E}^{10}, Z^{10}) = 2h_Z \; , \quad Hess(Z^{10}, Z^{10}) = 0$$

(here, we used the notation "$(\cdot)_Z$" to indicate the derivative $(\cdot)_Z = Z(\cdot)$ in the direction of Z) so that the matrix H of the components of $Hess(u)_z$ w.r.t. the frame $\{E^{10}, Z^{10}\}$ is

$$H = 2 \begin{pmatrix} 1 + 2hh_Z & h_Z \\ h_Z & 0 \end{pmatrix} \; .$$

Since the eigenvalues of H are $\lambda_\pm = 2\frac{(1+2hh_Z) \pm \sqrt{(1+2hh_Z)^2 + 4h_Z^2}}{2}$, we conclude that u is J-plurisubharmonic if and only if $h_Z \equiv 0$ and hence if and only if $h \equiv 0$ at all points (recall that, by assumptions, h vanishes identically around 0).

Therefore, we may construct arbitrary examples in which $u = \log \tau_o$ is not J-plurisubharmonic for almost complex structures J arbitrarily close to the standard one. This shows that in order to avoid situations like this it is not sufficient to restrict to a class of sufficiently small deformations of integrable structures. One needs some additional assumptions. One of them is the condition that J *is also "nice"*. In fact, one has the following result.

Theorem 6.8 *Let D be a nice circular domain with standard exhaustion $\tau_{(x_o)}$ and normal form $(\mathbb{B}^n, J)$. If J is a sufficiently small $\mathscr{C}^1$-deformation of J_{st}, then $u = \log \tau_{(x_o)}$ is the Green function with pole at x_o.*

Proof. We only need to show that $u = \log \|z\|^2$ is J-pseudoconvex on $\mathbb{B}^n \setminus \{0\}$. If $(\mathbb{B}^n, J)$ is nice, then $Hess(u)_z(\mathscr{Z}, \mathscr{H}) = 0$ at any $z \neq 0$. Since the spheres S_c are J-pseudoconvex for any J sufficiently close to the standard structure, the plurisubharmonicity of u follows directly by computing the Hessian along "orthogonal" directions. □

Putting all these facts together, one gets

Theorem 6.9 *Let D be an almost complex domain of circular type with center x_o in (M, J) strongly pseudoconvex. If the normal form $(\mathbb{B}^n, J')$ of (D, J) is very nice with J' sufficiently close to J_{st}, then*

a) the stationary foliation $\mathscr{F}^{(x_o)}$ consists of extremal disks w.r.t. Kobayashi metric;
b) the function $u = \log \tau_{(x_o)}$ is the almost pluricomplex Green function of D with pole x_o;
c) the distribution $\mathscr{Z}_z = \ker(Hess(u)_z)$ is integrable and the closures of its integral leaves are the disks in $\mathscr{F}^{(x_o)}$.

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